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Beyond ß: An Analysis of Alternative Risk Measures

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Beyond β: An Analysis of Alternative Risk Measures

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Abstract

Investors require a return from investing in stock securities that adequately compensate the investors for the risk level assumed. Therefore, any calculation of expected returns from a stock requires knowledge of the risk of the security. While there is no strong consensus on an ideal risk measure, traditionally risk has been conceptualized as volatility and is measured by the β of the stock or portfolio. This paper hypothesizes that alternative risk measures such as higher order moments, size, leverage, and price-to-book value add explanatory power to the β when predicting stock returns. Empirical analysis is conducted using both regression and portfolio methodologies and data collected on over 300 NYSE companies. The results demonstrate a clear lack of statistical significance of alternative risk measures in explaining returns and show that the relationship between returns and β for the time period 2003 – 2014 is negative. Additional testing is conducted by analyzing the impact of the financial crisis on the results and by changing market indices, neither of which significantly change the results obtained. This paper also builds a theoretical framework that may be used to model stock prices using a martingale process.

TABLE OF CONTENTS

I. Introduction

Stock markets are commonly known but are widely misunderstood. The general public often views the stock market in a manner similar to a casino, where fortunes are won and lost on nothing more than gambles (Geisst, 1982). Whether this is accurate depends on their respective probabilities of generating a return. At a casino, the various instruments of gambling are designed so as to slightly tilt the odds of winning in favor of the house. It is expected that given a long period of time, the house will eventually generate a return at the expense of its customers. The stock market is dissimilar to gambling in that there is no mechanism for tilting the odds of winning away from the investor. Figure 1.1 below shows the S&P 500 Index from January 1970 to January 2014. It is clear from the plot that over a long period of time, investors in the stock market have on average made positive returns on investments, despite occasional market downturns.

It is not enough to simply state that the odds of increasing wealth by investing money in stock securities are in the favor of the investor. In a casino the odds of winning may be calculated mathematically, and based on these odds the expected returns from placing bets may also be calculated. It is important to develop a framework by which the returns expected from investing in stock securities can be determined. This is no simple task. Since a purchaser of stock is concerned about the risks associated with investing money, high-risk stocks will be purchased only if they come with a correspondingly high

expected return. Therefore, the investor *requires* a higher rate of return if he/she is to purchase a high-risk security. To truly calculate understand how returns arise from stock investments, an adequate knowledge of the security's risk is required. A theoretical discussion of the return-risk relationship is developed and presented in Chapter II.

Figure 1.1: The S&P 500 Index, 1970 to 2014

As it is not always clear what an investor thinks about when he/she thinks about 'risk', the process of computing expected returns is challenging and requires an in-depth analysis. While it is difficult to ascertain the thinking pattern of an investor directly, this may be achieved using proxies that may indicate the nature of risk to an investor. For instance, if an investor is concerned about large market price fluctuations of the stock, then it is likely that a risk factor that measures volatility may be significant in explaining returns variations. Volatility has, in fact, long been considered as the quintessential measure of risk. Since Markowitz (1952), which utilized returns variance as a risk measure, the work of several researchers such as Treynor (1962), Sharpe (1964), and

Lintner (1965) has led to the development of the β measure as the key risk variable and the Capital Asset Pricing Model (CAPM) as the most common method of computing expected returns. More details on the β and CAPM may be found in Chapter III, which is a detailed theoretical discussion on risk.

It may be noted, however, that the β measure serves as a useful risk indicator only if the stock returns follow a distribution such as a normal distribution in which the mean and variance provide adequate information on the shape of the stock returns spread. Empirical literature such as Mandelbrot (2004) reveals that it may not be the case that returns are distributed normally, making it unlikely that the β captures all the risk information for the stock. As such, there is a need to develop alternative measures of risk if the returns are to be adequately explained.

This leads to the goal of this study: to analyze whether alternative risk measures are able to add predictive power to the β of a security when predicting the stock's returns. Chapter III also contains a theoretical analysis of the alternative risk measures utilized by this study. These include the downside β, skewness, excess kurtosis, size, leverage, and price-to-book ratio. The results of this analysis have important consequences for the financial world and for investors in particular. The β measure has conventionally been considered as the most important risk measure and academics and business practitioners alike often use it to gauge the risk of investing in a particular stock or portfolio. The findings of this paper may challenge these long-held investor beliefs and the importance of alternative risk measures as returns predictors may require a

shift in the way academics and investors gauge what returns should be expected when investing in a security.

It is also evident from the preceding discussion on stock returns that the normality of returns distributions plays a key role in determining the effectiveness of alternative risk measures. Several academics (Malkiel, 1996) still consider the change in stock prices to be a 'random walk', a situation where the prediction of stock prices is futile and there are no supernormal profits to be made. In this case, stock returns follow a normal distribution and the β measure serves as an adequate measure of risk. As such, Chapter IV provides a deeper understanding of the mathematical nature of stock prices as random walks and shows that random walks are a subset of a larger system of 'fair game' models called martingales. This allows future researchers to test stock price data to see if it matches with the properties predicted for a martingale process. Failure to match martingale properties results in the conclusion that stock prices may not be fair games (such as random walks). Chapter IV builds several concepts from measure theory to arrive at a mathematical discussion on martingale theory.

Following the theoretical framework, previous literature in this subject is reviewed in Chapter V so as to determine the optimal testing methodology, model specification, and data sample for conducting empirical analysis. The pioneering study of alternative risk measures was conducted by Fama & French (1992) which led to the Fama-French Three Factor Model. In this study the authors demonstrate that in addition to β, size and book-to-market ratio are important in explaining returns. To test their hypothesis use the cross sectional regression methodology developed by Fama &

MacBeth (1973), a study in which Fama-Macbeth empirically test the CAPM. The cross sectional regression methodology is adapted by this study for hypothesis testing as well. In addition, another methodology called the portfolio methodology is used for hypothesis testing. This method is a modification of a testing methodology developed by Black, Scholes and Jensen (2006). Prior empirical literature also provides motivation for the empirical work conducted by this study because prior empirical studies have mostly focused on specific risk measures and no one study has combined a large number of risk measures together to analyze how they perform when utilized in the same model.

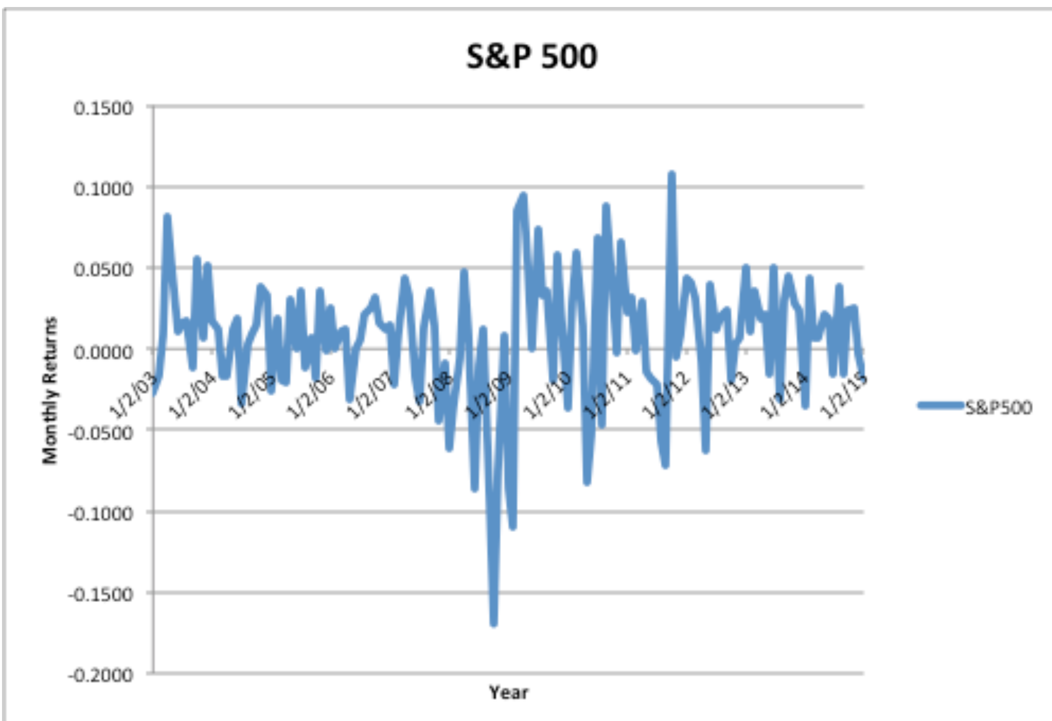

Figure 1.2: Monthly Returns on the S&P 500, 2003 to 2014

The empirical testing of this paper's hypothesis is presented and discussed in Chapter VI. Chapter VI begins with a discussion of the methodology and data selection for the empirical analysis and then progresses to present the results of the empirical

analysis. Note that the time period selected (2003 to 2014) contains a major structural break in the form of the 2008 financial crisis (often called the 'Great Recession'). This leads to a period of extremely volatile returns in the middle of the time sample as shown in Figure 1.2. This paper tests the effectiveness of alternative risk measures pre and post financial crisis to check if the financial crisis causes a change in the way investors perceive risk measures during an unstable economic climate. Following the empirical discussion, Chapter VII concludes this study.

II. The Return/Risk Relationship

As mentioned earlier in the introduction, the aim of this study is to analyze the effect of alternative risk measures on the returns of a portfolio. As a simplification, we assume that the only form of financial instruments under investment consideration is common stock. As such, a basic understanding of the economics behind shares and share pricing is important in understanding how traditional and alternative risk measures may affect the returns generated by a portfolio of these securities. This section will begin by first discussing the economics behind a stock in isolation. The paper will then progress to viewing stocks as part of a portfolio, which is a collection of financial instruments (or in our case simply stocks) put together for investment purposes. Section II.1 introduces stocks which is the main financial instrument pertaining to this study. In section II.2, the method behind calculating the value of stocks is established and finally, the return-risk relationship is determined in section II.3.

II.1. Introduction to Common Stock

A common stock share is simply an ownership share or equity interest in a corporation. Equity is the total amount of funds available to a company that have been provided by the owners (or stockholders) plus the retained earnings (or losses) of the corporation. Owners of equity in a corporation are legally entitled to a share of the company's profits that accumulate over time as shareholders' equity. This is why stocks

are also commonly referred to as *equity* or *shares.* The claim on the earnings of the corporation separates shareholders from bondholders and other creditors who lend the company money and are only entitled to receive interest payments and the return of the loan principal upon maturity. However, in the case of a corporate bankruptcy, the creditors must be paid their outstanding amounts before the legal owners are able to divide the residual assets of the corporation amongst themselves.

The stock exchange from where an investor may purchase shares of a particular public corporation is actually the *secondary market* for trading in stocks. It is secondary as the stocks are already in existence. The trade simply takes place between investors who are willing to buy and sell the shares of the corporation under consideration at the prevailing market price (Bailey, 2005). The *primary market* is where the corporation itself issues new shares and sells them to investors for the first time. Typically, primary markets are facilitated by underwriting groups consisting of investment banks that set the initial price range and oversee the sale of assets to investors.

Selling shares allows a company to accomplish two objectives: it is able to expand its capital base by utilizing the money it receives from the purchase of its stock and it is also able to better spread the risk that is associated with conducting business. For an investor, it provides a means for investing money that would have otherwise not been productively used. As such, an investor's objective is to obtain a return on his or her investment. Given this objective, it is clear that establishing a methodology for calculating the value of an equity instrument is critical. This valuation along with the investor's belief regarding the growth potential of the company under consideration is

crucial to the investor's ultimate investment decision. The next section discusses how to value an equity instrument and gauge whether the market has over or under-valued a security.

II.2. The Economic Value of Shares

It is important for an investor to know what the economic (or "intrinsic") value of a stock should be. The investor utilizes this information to decide whether or not the stock is currently overpriced or underpriced in the market. If the intrinsic value of the stock is lower than its market price, the market overvalues the stock and the investor will decide to sell the stock and if the intrinsic value of the stock is greater than the market price, the market undervalues the stock and the investor will choose to buy this equity. In terms of this study, understanding the components of the economic value of shares is crucial. This is because returns are changes in economic value and the aim is to explain returns. The next section presents a method of valuing stocks based on the principle that the value of a stock is the present value of all its future cash flows. The aim is to develop a mechanism by which a meaningful risk-return relationship may be determined.

II.2.1. The Dividend – Discount Model

The cash receipts associated with a common stock are (1) the dividends paid by the corporation per share, (2) the future earnings on each share, and (3) the future dividends for the holding period plus the price of the share at the end of that period.

Since any holder of stock in a corporation receives these dividends, if the corporation so chooses to pay them, the price of the stock in the market should be equal to the present value of all its dividend payments. To analyze how to calculate the price of a stock for a corporation in the current time period ($t = 0$), let us consider an arbitrary horizon, N and let us assume that the dividend payment in time period $t = i$ is D_i.

Since the aim is to develop a mechanism to identify the economic value of the stock to the investor, it is important to have a discount rate that adequately captures the level of risk involved in purchasing the stock. Since, this interest rate captures the amount an investor would have to earn to cover the risk associated with that particular stock, the rate we use to discount cash flows is called the investor's *required rate of return.* To calculate the present value of dividend payments, let the investor's required rate of return for holding the stock for a time period t be r. It is assumed that this rate remains constant. Now consider a corporation Y who's shares are to be valued. Let P_0 be the current economic value of the share to a one-year investor X. Since X will sell the share in one year, P_0 is calculated as:

$$P_0 = \frac{D_1}{1+r} + \frac{P_1}{1+r}$$

If X invests in corporation Y for two years,

$$P_0 = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \frac{P_2}{(1+r)^2}$$

In general, if X invests for N years,

$$P_0 = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \frac{D_3}{(1+r)^3} + \cdots + \frac{D_N}{(1+r)^N} + \frac{P_N}{(1+r)^N}$$

In the case that the firm exists forever and continually pays dividends, we can let $N \to \infty$. In this case, the value at time $t = 0$ is,

$$P_0 = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \frac{D_3}{(1+r)^3} + \cdots = \sum_{i=1}^{\infty} \frac{D_i}{(1+r)^i} \tag{2.1}$$

To reach a simple valuation equation, we first consider the case of a constant dividend payment D that does not change over time. Then the value at time $t = 0$ is,

$$P_0 = \sum_{i=1}^{\infty} \frac{D_i}{(1+r)^i} = \sum_{i=1}^{\infty} \frac{D}{(1+r)^i}$$

This takes the form of a perpetuity (which is a geometric series) and using the formula for the summation of a geometric series, we obtain,

$$P_0 = \sum_{i=1}^{\infty} \frac{D}{(1+r)^i} = \frac{D}{r} \tag{2.2}$$

To assume a constant dividend payment is somewhat unrealistic. Firms typically grow and as the earnings of a firm increase, so do their dividend payments. Since it is difficult to determine when a corporation’s dividends will increase, a commonly made assumption is that in the long-run the growth rate becomes constant. So we assume a constant growth rate g for the dividend payments. We assume that the dividend paid in the first time period $t = 1$ is D. Then for the time period $t = 2$, the dividend payment is $D(1+g)$; for $t = 3$, the dividend is $D(1+g)^2$, and in general for any time period $t = i$, the dividend is $D(1+g)^{i-1}$. Assuming a holding of an infinite time period in the corporation Y, the value at time $t = 0$ to an investor X may now be computed as,

$$P_0 = \frac{D}{1+r} + \frac{D(1+g)}{(1+r)^2} + \frac{D(1+g)^2}{(1+r)^3} + \cdots = \sum_{i=1}^{\infty} \frac{D(1+g)^{i-1}}{(1+r)^i}$$

This takes the form a growing perpetuity and finding the sum of this geometric progression gives us the ***Gordon Growth Model*** (Gordon & Shapiro, 1956) valuation formula shown below:

$$P_0 = \frac{D}{r - g} \tag{2.3}$$

where D is the dividend payment at the end of the first time period, r is the investor's required rate of return, g is the growth rate of the dividend payments, and $g < r$.

To understand how to compare the intrinsic value to the market price of the stock, it is important to first understand the concepts of *arbitrage* and the *Law of One Price*. Arbitrage is the method of accruing abnormal profits by buying equivalent goods in separate markets to take advantage of an existing price difference. In general, an *arbitrage opportunity* is the existence of a situation in which it is possible to make profits without taking any risk or making an investment. Since the net present value of any arbitrage opportunity would be greater than zero, investors would clearly wish to take advantage of it. While the initial investors who are able to trade quickly may be able to exploit the arbitrage opportunity, prices will soon react to the trades being made and prices will equalize, eliminating the arbitrage opportunity. As such, in *normal market conditions*, arbitrage opportunities do not exist. This leads to the Law of One Price, which states that if equivalent investment opportunities trade at the same time in different competitive markets, then the price for the trade must be the same in all markets (Berk & DeMarzo, 2014).

The Law of One Price is an important consideration when attempting to value an asset. The price of a stock in a market must be such that it would not be possible to

make profits by simply arbitraging between different equivalent stocks or equivalent investment opportunities without any net investment or accepting any risk (Houthakker, 1996). Since the price of equivalent investment opportunities must be the same in all markets, it is possible to value a financial instrument such as a stock by simply finding an equivalent investment opportunity whose value is already known. This equivalent investment opportunity must generate the same set of cash flows as the stock we have in question. To prevent arbitrage from occurring, both these investment opportunities must be equally valued. Therefore,

Economic Value [Stock] = Present Value [All Cash Flows Paid by Stock] = Market Price

This is called the *no-arbitrage price* of the stock. The calculated P_0 is actually the economic value of the stock and the no-arbitrage price. However, it is questionable whether this no-arbitrage price for the stock is actually realized in the market owing to factors such as price stickiness and transactions costs. Since returns on stocks are generated through dividend payments and capital gains (i.e. increase in the market price of the stock), knowledge of the stock's economic value is not as crucial as knowing whether or not this economic value is equal to the price of the share in a competitive market. The price of a stock is equal to its economic value only if markets operate *efficiently*. Market efficiency and the validity of the efficient markets hypothesis are discussed in the next section.

II.2.2 The Efficient Markets Hypothesis

The outcome of a perfectly competitive market is that all participants in market trades are price takers. No one actor can influence the price of a commodity in a perfectly competitive market. The three main characteristics of a perfectly competitive market are: (1) a large number of buyers and sellers (i.e. negligible barriers to entry), (2) a homogenous product, and (3) perfect information between buyers and sellers. The necessary outcome of these assumptions about the market is that all participants are price takers. Consider the market for equity to see if it can be classified as a perfectly competitive market. Due to the advent of the Internet, a greater number of people now have access to the stock market suggesting that there are a large number of buyers and sellers of stock. While there are transactions costs to buying and selling shares, the ease with which these transactions can now be made makes the transactions costs relatively low. Clearly the shares themselves are homogenous goods. There is no difference between any two common stocks from the same corporation and both represent an equal ownership share in the company. The third assumption regarding perfectly competitive markets is debatable and has often been the subject of question for academics as well as investors looking to beat the market consistently.

An implication of the stock market being perfectly competitive is that no investor will be able to make above normal profits in the long - run, i.e. no investor is expected to consistently beat the market. In essence, this means that investors will not be able to consistently identify under or over valued assets and take advantage of this arbitrage

opportunity by buying low and selling high. In financial terms, when such opportunities are unavailable it is said that markets are *efficient*. Stock markets are said to be efficient if the price of the financial instrument under consideration adjusts quickly and completely to new information. Depending on how much the market price takes into account the information that exists gives a measure of how efficient the market for equity is. The three forms of market efficiency are:

- Weak Market Efficiency: Prices reflect all available past information. This suggests that *fundamental analysis* may produce superior investment results but not *technical analysis*. Fundamental analysis is a technique wherein an investor judges whether current market prices under or over value the real economic value of an asset and make investment decisions based on this judgment. Technical analysts aim to identify patterns and trends in historical stock price data and use these patterns to predict future stock prices.
- Semi-Strong Market Efficiency: The market price of a stock fully reflects all of the known past and present public information regarding the company. This includes the firm's past history and any information that may be learned by analyzing the firm's financial statements, its industrial information, and the general economic climate. Under this form of market efficiency, either form of stock price analysis does not consistently provide superior investment results. However, since the price of the stock does not incorporate any known *private* corporate information, investors with access to inside information may achieve consistent

above normal investment returns. This is known as *insider trading* and the use of inside information for private gain is illegal.

- Strong Market Efficiency: The market price of the stock represents all known public and private information. Even individuals with access to inside information cannot consistently be expected to make excess returns on investments (Mayo, 2011).

Since corporate information is available publicly, securities markets are highly competitive, and transactions costs are low, many academics believe that the efficient markets hypothesis holds for securities markets. Note that it is not the case that it is impossible for stock traders to make a profit from utilizing technical analysis. Instead, the efficient markets hypothesis suggests that such profits from the existence of trends and patterns will be competed away and the ability of traders to consistently make supernormal profits will not be expected in the long run. As such, price changes from any point in time should be independent of any previous historical information or data, i.e., if markets are efficient, then price changes are a *random walk* (Cootner, 1967). A random walk simply implies that price changes are unpredictable and that any patterns that form are purely accidental. Some empirical studies have also demonstrated that the outcome from a random walk time series actually resemble the progression of stock prices (Roberts, 1959).

According to Cootner, the reason for efficient markets is the separation between *informed* and *uninformed* investors. The uninformed investors invest in stocks without much information and insight and are vulnerable to buy or sell based on rumors, often

over (or under) reacting. In such a case, the market may over or under price the asset allowing for an arbitrage opportunity. The informed investors view this as a buy signal if the market underprices the asset or a sell signal if the market overvalues it. The actions of the informed investors bring the price back to the underlying economic value of the asset, thereby nullifying the arbitrage opportunity. Since the time period for capitalizing on the arbitrage opportunity is small, it is unlikely that the same set of informed investors will be able to act quickly enough to always take advantage of the price aberrations brought about by the uninformed investors. As such, no particular set of informed investors would be expected to beat the market on a consistent basis. While some informed investors might be able to make profits from arbitrage opportunities, they are not expected to consistently outperform even the uninformed investors.

Despite the statistical and theoretical evidence for the efficient markets hypothesis and its random walk implications, there are economists and professional investors who emphasize the psychological and behavioral elements that play a part in determining stock prices. They also believe that future stock prices are to some extent predictable using historical data as well as the valuation techniques used by fundamental analysts. While it is true that certain stock market anomalies do exist, it is not necessary that such anomalous price behavior provides investors with a portfolio trading opportunity that earns extraordinary returns adjusted for risk (Malkiel, 2003).

II.3. The Return-Risk Relationship

Since the aim of this study is to examine the impact of alternative risk measures on the returns generated by a portfolio of shares, it is important to understand the relationship between returns and risk. Positive returns from a single stock arise from holding the stock for a time period such that the market price for the stock is higher than the purchase price. If the efficient markets hypothesis is accepted, then the returns are actually the result of a change in the intrinsic value of the share, which is reflected by its market price. For instance, for a one-year investor, the return R_t at time $t = 1$ is given by:

$$R_1 = \frac{P_1 + D_1}{P_0} - 1 \tag{2.4}$$

As such, under the assumption of efficient markets,

$$\text{Returns} = \text{f}\ [\text{Change in Economic Value}]$$

It is clear from this relationship that to analyze the effect of risk on the returns of a stock, it is important to look more deeply at the stock's economic value and what determines it. The Gordon Growth Model presents the key determinants of a share's economic value. Based on equation 2.3, the economic value of a share is dependent on the following three variables:

$$\text{Economic Value} = f\ [D, g, r]$$

where D is the dividend payment, g is the growth rate and r is the investor's required rate of return. The dividend paid by a firm to its shareholders is taken from the firm's earnings, as each common stock represents an ownership share in the company and a

claim on the firm's earnings. The dividend in time period t must then be a proportion b_t of the earnings Y_t of the firm, i.e.,

$$D_t = b_t\, Y_t \tag{2.5}$$

If we assume a constant rate b of earnings payments as dividends, then the dividend payment at time period t is

$$D_t = b\, Y_t$$

Since, the earnings are increasing at a rate g,

$$Y_t = g\, Y_{t-1}$$

$$\therefore D_t = b\, g\, Y_{t-1} \tag{2.6}$$

This leads to the following dependence for the economic value of a share:

$$\text{Economic Value} = \text{f}\,[\text{Earnings, Earnings Growth}, r]$$

To understand the role of risk in determining the economic value of the share, the investor's required rate of return must be analyzed. Since there are several firm-specific and market-specific risks associated with investing in equity, the required rate of return (r) would have to be higher than the *risk-free interest rate*. Each new shareholder's eventual risk is equal to the price he or she may pay for the number of shares purchased. This risk is essentially a *market risk*, i.e., the amount this investment may be affected by the change in price of the corporation's shares in the stock market.

For a corporation's shares to appeal to an investor, several factors must be taken into consideration. Firstly, the investor must believe that investing in the corporation under consideration will result in a return that is higher than investing the same amount in other shares or some other alternative financial instrument such as bonds,

commodities, real estate, etc. Secondly, regardless of the growth potential of the corporation itself, an investor must believe that the market and economic conditions are conducive for equity appreciation. This is because the investor is concerned with the risk that is always associated with trading in equity. In a recessing economy, the price of stock falls due to low investor confidence. In such an economic situation an investor may be hesitant to buy equity, afraid of a further decline in prices. Lastly, both the corporation and the investor must understand these factors in order to use the prevailing market conditions to their mutual advantage (Geisst, 1982). This need for risk compensation increases the investor's required rate of return above the risk-free interest rate i. Therefore,

$$r = \text{f}\,[i, \text{Risk}]$$

Since the investor's required rate of return depends on the risk associated with investing in the particular share and the economic value further depends on the investor's required rate of return, then it follows that the economic value of a share depends on the risk. Since returns on a stock arise from changes in its economic value, it is clear that,

$$\text{Return} = \text{f}\,[\text{Risk}]$$

This result leads into the discussion on risk, which is the key aspect of this study. Risk from the perspective of modern portfolio theory along with alternative methods of conceptualizing risk are highlighted in the next chapter.

III. Modern Portfolio Theory and Alternative Risk Measures

Choices are key to every human individual, whether the choices are trivial, or are of a more important nature such as choices with relation to education, family, or livelihood. Attached to each choice an individual makes, there may be some aspect of *gain* as well as an element of *risk*. While an individual hopes to maximize the return and eliminate the risk from his actions at the same time, the fact that we cannot always perfectly foresee all the consequences of our actions means that risk forms a key part of every decision, no matter how carefully thought out and calculated. Indeed in most situations risky gambles have the potential to provide the risk-taker with high levels of returns, provided chance works in favor of the risk-taker. We measure this 'return' on human choices as *utility*.

Risk is typically viewed as uncertainty about the future that is both measurable and actionable. It may be of two varieties: (i) Pure risk where maintaining the status quo is the best-case scenario, and (ii) Speculative risk which involves a gamble, i.e. an aspect of winning or losing. Situations such as a person's house burning down from a fire or the unexpected death of a family member are examples of pure risk scenarios, whereas participating in a lottery is an instance of a speculative risk scenario. Investing in common stock is a form of speculative risk.

This chapter serves as an in-depth explanation of risk. In section III.1, the theoretical framework behind risk and the Friedman-Savage Utility Model is developed.

Section III.2 presents risk as viewed from a Modern Portfolio Theory perspective and alternative methods to conceptualize risk are developed in section III.3.

III.1 The Theoretical Framework for Risk

When presented with choices involving risk, a consumer unit faces two options: gambles and insurance. Depending on the consumer's inherent risk-averse or risk-seeking nature, a consumer unit chooses to purchase a gamble or to purchase insurance against the risk. This section presents the theoretical framework that drives consumers to purchase gambles/insurance. The theoretical model presented for determining utility choices under risk and uncertainty is the Friedman-Savage utility choice model that is discussed in subsection III.1.1. In subsection III.1.2, the paper presents different measures of calculating risk aversion since risk aversion is central to the notion of making choices under risk. The decision to buy insurance/gambles rests heavily on a consumer unit's risk aversion and subsection III.1.2 discusses mathematical evaluations of this risk aversion.

III.1.1 The Friedman-Savage Utility Choice Model

The concept of utility stems from the Rational Actor Model as proposed by John von Neumann in the vastly influential book: *Theory of Games and Economic Behavior*. Neumann provides axioms that he considers to be true for every rational actor (the decision making individual) when exhibiting a set of preferences. For notational purposes let X refer to the rational decision-making actor. The following are the axioms that summarize X's preferences:

1. The set of preferences by which an individual chooses is complete and consistent, i.e., if there are two objects (or combinations of objects) a and b, then there are three ways in which a and b may be ordered by preference: $a > b$, $b > a$ or $a = b$.
2. Preferences are transitive, i.e., if $a > b$ and $b > c$, then it must be the case that $a > c$.
3. If $a \leq b \leq c$, then there exists a probability $p \in [0,1]$ such that $pa + (1 - p)c = b$.
4. Sets of preferences are independent from one another so if $a < b$, then for any c and $p \in [0,1]$, $pa + (1 - p)c < pb + (1 - p)c$.

Under the assumption of these four axioms, von Neumann shows that each rational actor has a utility function that maps a particular choice to a distinct numerical utility value (Von Neumann, 2007). It now makes sense for us to talk about individuals and their choices in terms of utility-maximizing behavior.

Now assume that X represents a consumer unit (either a family or household). It is now possible to develop a theory as to how X chooses among alternatives available to it. The model used to determine consumer choice from a set of alternatives is the Friedman-Savage utility model (Friedman & Savage, 1948). In general, when choosing among alternatives a consumer unit (X) behaves as if:

- It has a set of preferences that are consistent and complete.
- These preferences may be perfectly described by a *utility function* that maps each alternative to a corresponding numerical value called its utility.

- Its aim is to maximize its expected utility.

If it is to be assumed that risk does not exist amongst a certain set of alternatives, then the choice of the consumer unit is simple: X chooses that alternative from which it derives the highest utility. However, as mentioned above, risk plays a vital role in choice-making analysis. In fact, in almost all economic choice situations, consumer units can be thought to be making a choice between insurance and gambles. Insurance is coverage against risk whereas gambles are acceptances of risk. In an economic framework, an individual may choose to invest in securities, thus taking a gamble in order to gain a higher monetary reward whereas in another case, the consumer unit is willing to sacrifice a portion of his wealth so as to insure himself against the risk of losing his house, or having an accident. As such, consumer units display *risk-seeking* (or *risk-loving*) as well as *risk-averse* behavior depending on their natural tendencies. Thus, when choosing amongst a set of alternatives that include risk, X still aims to maximize the expected utility derived from these alternatives.

To simplify the choice analysis under risk, this paper assumes that all the alternatives open to a consumer unit may be expressed completely in monetary terms. This allows for a simultaneous analysis of a wide variety of qualitatively different alternatives as well as provides a common variable when trying to construct a utility function. Consider total utility U to be a function of X's wealth alone. Let w represent the wealth of a consumer unit per unit time; then $U(w)$ is the utility attached to that wealth level if regarded as certain. Since X must choose between different alternatives to calculate his wealth, w is a random variable. Alternatives may be now considered as

possible wealth levels: w_1, w_2, etc. and the Friedman-Savage hypothesis implies that X would choose the wealth to which it attaches the most utility. It may be assumed that utility of monetary income increases as wealth increases. This assumption must be made in order to rationalize the obvious decision of the highest wealth made amongst riskless choices under this theoretical framework.

Alternatives that include risk comprise of probability distributions of multiple wealth levels. For the purposes of constructing the theory, let there exist a two alternative scenario: an alternative including a risk (gamble) called A and a riskless alternative B. The choice set B consists of a certain wealth level of w_0. The choice set A consists of a chance (probability), p of getting an income w_1 and a chance $(1-p)$ of obtaining a wealth level w_2. It is assumed that $w_2 > w_1$. The utility of alternative B is simply the utility of the certain wealth level: $U(w_0)$. The expected utility of A may be calculated as: $\overline{U}(A) = E[U(A)] = pU(w_1) + (1-p)U(w_2)$. A consumer would choose A if $\overline{U} > U(w_0)$ and will choose B if $\overline{U} < U(w_0)$.

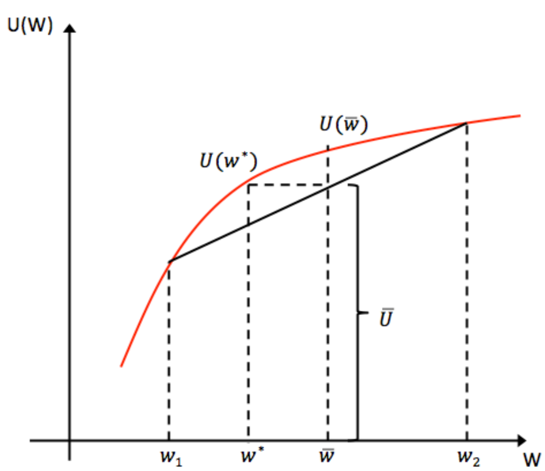

Figure 3.1(a): Risk-averse investor

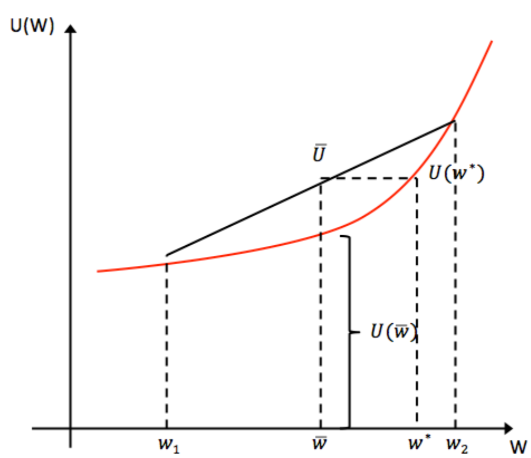

Figure 3.1(b) Risk-seeking investor

The expected wealth from A is calculated as: $\overline{w}(A) = E[w(A)] = pw_1 + (1-p)w_2$. If $w_0 = \overline{w}$, the gamble is considered to be *fair*, since the expected value from either alternative is the same. Given these circumstances, X's attitude towards risks may be determined. If X chooses A, it shows a preference for risk, i.e. $\overline{U} > U(\overline{w})$ and $\overline{U} - U(\overline{w})$ may be viewed as the utility X attaches to this risk. If under the same circumstances, X were to choose alternative B, it may be said that X prefers certainty, i.e. $\overline{U} < U(\overline{w})$ and $U(\overline{w}) - \overline{U}$ is the value it attaches to this certainty. If X is indifferent between choices A and B, then $\overline{U} = U(\overline{w})$. We define the wealth equivalent, w^* as the wealth level that provides to X the same level of utility as the gamble A, i.e. $U(w^*) = \overline{U}$. From our assumption that utility increases with wealth, if $\overline{U} > U(w^*)$, then it must be the case that $w^* > \overline{w}$. The same extends for $\overline{U} \leq U(w^*)$. To illustrate the cases of risk-averse and risk-seeking behavior, consider the graphs provided in Figure 3.1. Fig. 3.1(a) represents a typical utility curve for a risk-averse consumer unit ($\overline{w} > w^*$) whereas Fig. 3.1(b) represents the utility function of a risk-seeking consumer unit ($\overline{w} < w^*$). $\overline{w}$ is represented on the plot by a point that divides the interval $[w_1, w_2]$ in the proportion $\frac{1-p}{p}$.

In Fig. 3.1(a), the utility curve is drawn so as to make $\overline{w} > w^*$. This implies that if X has a choice between A and a fixed income alternative (B) with $w_0 > w^*$, it will choose B. For a fixed wealth $w^* < w_0 < \overline{w}$, X is paying $\overline{w} - w_0$ for certainty, i.e. X is *buying insurance*. For $w_0 > \overline{w}$, X is being paid for certainty in the amount of $w_0 - \overline{w}$ and we say that X is *selling a gamble*. If w_0 is lower than w^*, then the consumer unit will prefer A to B since he has to pay more for certainty than the maximum amount

($\overline{w} - w^*$) that he is willing to pay. The price of insurance has become so high that X finds it better to gamble than settle for a certain wealth level.

In Fig. 3.1(b), the utility curve is drawn so as to make $\overline{w} < w^*$. This implies that if X has a choice between A and a fixed income alternative (B) with $w_0 < w^*$, it will choose A. For a fixed wealth $w^* > w_0 > \overline{w}$, X is paying $w_0 - \overline{w}$ to gamble, i.e. X is *buying a gamble*. For $w_0 < \overline{w}$, X is being paid for uncertainty in the amount of $\overline{w} - w_0$ and we say that X is *selling insurance*. If w_0 is higher than w^*, then the consumer unit will prefer B to A since he has to pay more to gamble than the maximum amount ($w^* - \overline{w}$) that he is willing to pay. The price of gambling has become so high that X finds it better to take the safe certain alternative gamble than gamble on A.

The issue with conceiving of utility functions in this manner is that we are unable to explain risk-averse as well as risk-seeking behavior from the same individual. As mentioned earlier, people tend to invest their money to allow for more wealth in the future as well as spend money on insurance against medical and other risks. With utility functions as described in Figure 3.1, this kind of behavior is inexplicable. Friedman and Savage list 5 observed behavior traits that they aim to explain using a suitably designed utility function: (a) consumer units prefer larger incomes to smaller incomes (b) low-wealth consumer units buy (or are willing to buy) insurance (c) low-wealth consumer units also buy (or are willing to buy) lottery tickets (d) a large number of low wealth level consumer units buy (or are willing to buy) both insurance and lotteries, and (e) lotteries tend to have more than just one prize (Friedman & Savage, 1948). The proposed utility function is plotted in Figure 3.2. The explanation as to why this utility

function satisfies the above requirements has been omitted since it goes beyond the purview of this paper.

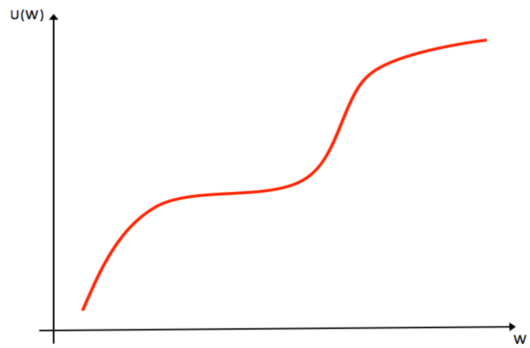

Figure 3.2: Utility curve for a typical consumer unit (Friedman-Savage)

It is possible to interpret the utility function in Figure 3.2 by considering the two concave segments of Figure 3.2's utility function as corresponding to two qualitatively different socioeconomic levels. The convex segment may then be interpreted as a transition period between two levels of income. An increase in wealth that raises X's relative position within its current socioeconomic level yields diminishing marginal utility. Increases in wealth that take X to a new socioeconomic status yield increasing marginal utility. A low income consumer unit would prefer a certain income to a gamble that at best may make him one of the most prosperous in his class or at worst make him one of the least prosperous in this class. However, the same consumer unit will be willing to accept a gamble that offers even a small chance to take him out of his current class and into the *middle* or *upper* socioeconomic levels, despite the existence of a large chance of moving slightly lower in his current low income group. Figure 3.2 represents only two such socioeconomic classes with one interim (risk-seeking) period. There may be several such class divisions, which would simply involve the addition of more

alternating convex and concave segments. The middle convex segment is not regarded as an economic class itself since it is not expected that consumer units will spend much time in this interim phase. As risk-seekers, they will either win lotteries and progress to the higher socioeconomic strata or lose gambles and fall back to the prior socioeconomic group.

III.1.2 Measuring Risk Aversion

Having understood how a rational consumer unit chooses between two or more alternatives that may include risk, we must now understand how to measure risk-aversion. Qualitatively, risk-aversion corresponds to the concave portions of the graph presented in Figure 3.2 but it is still unclear as to how to quantify risk-aversion (or in the opposite sense, risk-seeking). Kenneth Arrow, in *The Theory of Risk Aversion*, defines a risk averter as "one who, starting from a position of certainty, is unwilling to take a bet which is actuarially fair" (Arrow, 1971). To quantify this risk-aversion, we must first assume the utility function to be twice differentiable. We now have: $U'(w) =$ marginal utility of wealth, and $U''(w) =$ rate of change of marginal utility with respect to wealth. Since we have already assumed that utility increases as wealth increases (i.e. $U(w)$ is a strictly increasing function), we know that $U'(w) > 0$ for all w. To avoid contradiction, we must also assume $U(w)$ to be *bounded*, i.e. $\lim_{w\to 0} U(w)$ and $\lim_{w\to\infty} U(w)$ exist and are finite. Economic theory as well as empirical observations imply that people tend to be more risk-averse than risk-seeking which means that $U'(w)$ is eventually a strictly

decreasing function i.e. as w increases beyond a certain point, diminishing marginal utility of wealth is observed.

Arrow suggests two measures for quantifying risk aversion. The following are the measures for *absolute* and *relative* risk aversion:

$$\text{Absolute risk aversion} = R_A(w) = -\frac{U''(w)}{U'(w)} \tag{3.1}$$

$$\text{Relative risk aversion} = R_R(w) = -w\frac{U''(w)}{U'(w)} = w\,R_A(w) \tag{3.2}$$

It is now important to interpret the meaning of these risk aversion coefficients. It must first be noted that both measures are positive when X exhibits risk aversion ($U''(w) < 0$). The relative risk aversion represents the elasticity of the marginal utility of wealth for consumer unit X. It remains unchanged with respect to changes in the units of utility or by changes in the units of wealth. The measures also have important behavioral interpretations. To understand these implications, consider a consumer unit X that has a current wealth W. X is offered a gamble of either winning or losing an amount h with associated probabilities p and $(1 - p)$ respectively. X will accept the gamble for sufficiently large values of p and reject it if p is small (if X is a risk averter, X will refuse all gambles with $p \leq 0.5$). This willingness to accept or reject bets will also be dependent on W, the current level of X's wealth. Given a particular bet amount h and a level of wealth W, there will be a probability $p(W, h)$ that makes X indifferent between acceptance and refusal. If we fix our attention to small values of h, $p(W, h)$ may be approximated by a linear function as follows:

$$p(W,h) = \frac{1}{2} + \frac{R_A(W)}{4}\, h + (\text{terms of higher order in } h) \tag{3.3}$$

From this we can see that the absolute risk aversion $R_A(w)$ measures the insistence of a consumer unit for better-than-fair odds (at least when the bets are small). When we measure the bets in terms of proportions of X's wealth W instead of absolute terms, the relative risk aversion replaces the absolute risk aversion. The amount of the bet is now denoted by nW where n is the fraction of wealth that is gambled. By substituting for $h = nW$ in eqn. 3.3, and using the definitions of risk aversion measures from (4) and (5), we have:

$$p(W,nW) = \frac{1}{2} + \frac{R_R(W)}{4}\, n + (\text{terms of higher order in } n) \tag{3.4}$$

Pratt (1964) offers his own interpretation of absolute and relative risk aversion measures. Let X be faced with a random income W and an alternative fixed income W_0. If X is a risk averse decision maker, then X is willing to accept a wealth level W_0 below the expected value of the gamble, $E[W]$. The difference (as mentioned earlier) is the insurance premium. Choose W_0 such that X is just indifferent between W_0 and the random income level W. We define the insurance premium $\pi = E[W] - W_0$. If this distribution is concentrated sufficiently (mathematically, if the third absolute central moment is sufficiently small in comparison to the variance σ^2), Pratt shows:

$$\pi = \left(\frac{1}{2}\right) \sigma^2\, R_A(W_0) + \, terms\ of\ higher\ order \tag{3.5}$$

Therefore, the absolute risk aversion, $\mathrm{R_A(w)}$ is approximately twice the risk/insurance premium per unit of variance for small gambles (Pratt, 1964). The importance of an investor's risk aversion is displayed in his/her required risk premium.

The more risk averse an investor is, the higher will be the investor's required rate of return on investing in equity. To maintain arbitrage free pricing, each individual must be compensated for the risk he/she is willing to take and this compensation rate is determined by the investor's risk aversion. Since it is hard to map a utility choice function for investors, thereby making it hard to accurately calculate an investor's risk aversion, alternate methods exist to determine the investor's required rate of return. This method is explored further in this chapter.

III.2. Risk, Volatility and β

So far in this chapter the theoretical framework behind the conception of risk has been established. This section now turns to risk as viewed from a financial perspective and in particular how risk pertains to shares and the stock market. It has already been shown that an investor in common stock requires a rate of return that is higher than the risk-free rate. The difference between the investor's required return on investing in stock and the risk-free rate is called the *risk premium*, or the excess return an investor requires as an incentive to purchase a particular stock. It is this risk premium and its constituency that comprises the key aspect of this section, which builds on the return-risk relationship. In subsection III.2.1, we discuss the constitution of risk for an individual stock versus a large portfolio. Subsection III.2.2 introduces the concept of *beta* and builds on this notion as the key component of risk for a stock or a portfolio. Finally, in subsection III.2.3, the market risk concept of β is critiqued, which paves the way for the use of alternative risk measures.

III.2.1 Risk – Large Portfolios vs. Individual Stocks

Figure 3.3 below plots the historical realized returns against the historical volatility of large portfolios. A simple linear relationship is observed between the realized returns and the historical volatility. More specifically, large portfolios that had a higher volatility also had a higher return over the time period. This is an intuitively obvious result: investors in riskier stock will require a higher expected return; their risk premium should be higher.

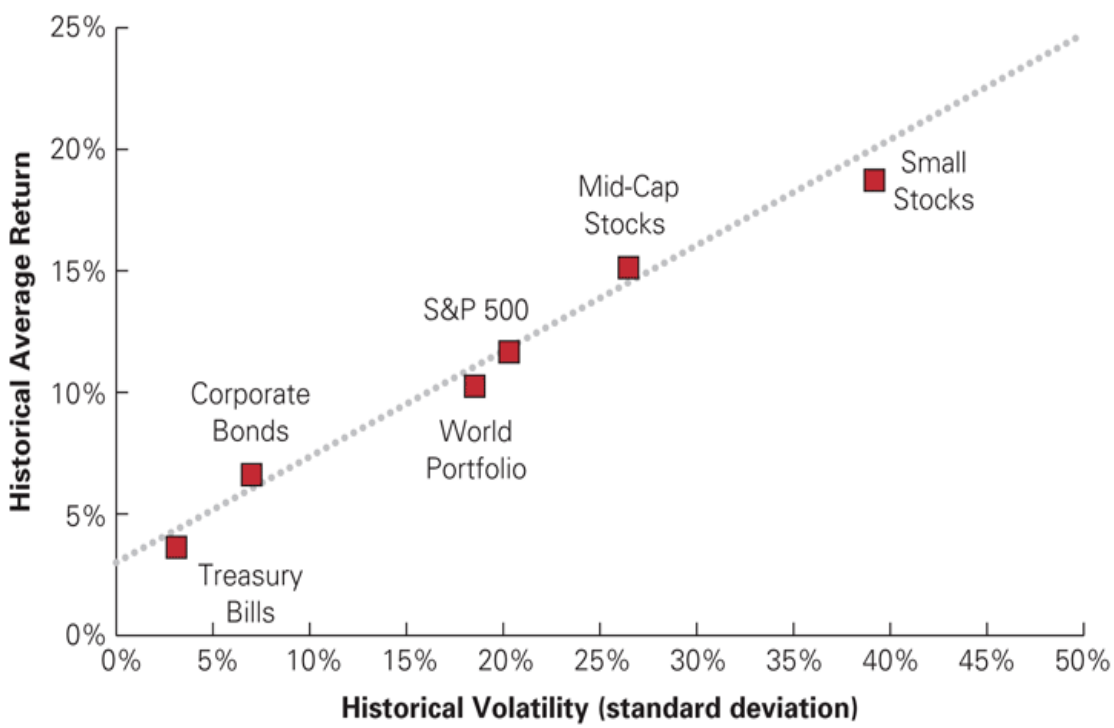

Figure 3.3: Plot of Historical Avg. Returns against SD for large portfolios[1]

Before the volatility of stock is further explored, a quick recap of some basic probability formulas may be useful. The *expected* or *mean return* $E[R]$ from an

[1] (Berk & DeMarzo, 2014)

investment project is the average of the possible returns, weighted by the probability of each outcome. If there are n possible outcomes and p_i represents the probability of occurrence, where $i = 1 \dots n$, then

$$E[R] = \sum_{i=1}^{N} p_i \, R_i \tag{3.6}$$

Two common risk measures of an investment are its *variance* and *standard deviation.* The variance $Var[R]$ of the returns generated from a probability distribution is the expected squared deviation from the mean. We calculate this as:

$$Var[R] = E[(R - E[R])^2] = \sum_{i=1}^{N} p_i \, (R_i - E[R])^2 \tag{3.7}$$

The standard deviation $SD[R]$, is another measure of risk and is computed as:

$$SD[R] = \sqrt{Var[R]} \tag{3.8}$$

While both the variance and standard deviation measure the extent to which the probability distribution is 'spread out', the standard deviation is simpler to interpret as it has the same units as the returns. The standard deviation is a financial measure of volatility.

It is hard and often highly inaccurate to estimate all the possible outcomes from an investment and assign to each of them an accurate probability. It is better to estimate the expected return by using the arithmetic mean return of the historical returns from time $t = 1 \dots N$ as follows:

$$\bar{R} = \frac{1}{N} \sum_{t=1}^{N} R_t \tag{3.9}$$

It is also possible to estimate the variance and standard deviation from the historical realized returns:

$$Var[R] = \frac{1}{N-1}\sum_{t=1}^{N}(R_t - \bar{R})^2 \tag{3.10}$$

$$SD[R] = \sqrt{Var[R]} \tag{3.11}$$

It may be noted that an assumption made when using historical returns is that the stochastic process of returns that are generated are independent and identically distributed. This is a fair assumption given the claim of market efficiency, which allows us to think of returns as having been generated from a normal distribution with the same expected value and variance as well as being independent. Since this process requires estimating the expected values, there is an error of estimation associated with using historical returns. This is measured simply as the standard deviation of the independent and identically distributed historical returns.

$$SE = SD[\bar{R}] = \frac{SD[R_t]}{\sqrt{N}} \tag{3.12}$$

Using the same plot as Figure 3.3 but adding to it the top 500 US stocks by size gives us Figure 3.4 below. It is clear from the graph that the linear relationship between returns and volatility that exists for large portfolios does not hold true for individual stocks. While it is evident that large stocks on average tend to have lower volatility than smaller stocks, each of these large stocks still has a higher volatility than a portfolio of large stocks such as the S&P 500. There is no discernable pattern for small stocks as is

clearly evident from the graph. Since the return-volatility trade-off no longer holds for individual stocks, we delve further into the risk constituency for stocks.

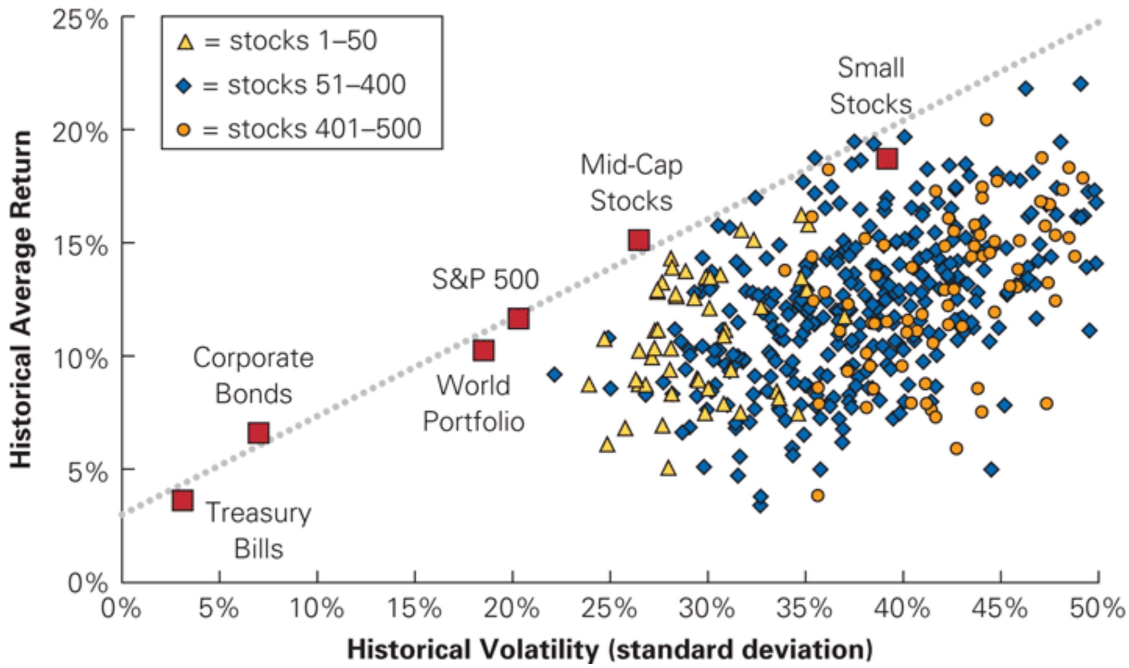

Figure 3.4: Plot of historical avg. returns against SD for large portfolios and 500 stocks[2]

Every investor that purchases an isolated stock is subject to two forms of risk – risk *individual* to that firm itself, and risk faced by all possible investments or risk that is *common* to all available stock investments. Individual risk for a stock arises from individual firm characteristics and is isolated to the individual firms themselves. This is called *firm-specific, idiosyncratic,* or *unique* risk. Firms are also subject to the broader market conditions and an investor always faces the risk of market fluctuations. This risk is called *market or systematic* risk.

If a few stocks that are included in a large portfolio suffer firm-specific losses, there are other stocks that can counterbalance these losses. It is unlikely that all or even

[2] (Berk & DeMarzo, 2014)

many of the stocks will move together due to firm-specific risks. These risks are uncorrelated with each other and are therefore *independent* of one another. These risks are *diversifiable* from the perspective of an investor and by creating a large portfolio it is possible to significantly reduce the firm-specific risks. Along with idiosyncratic risks, the firms are also subject to the larger economy in general and are affected by market fluctuations. Since a market fluctuation affects all firms in the economy, the stocks all move up or down in unison, preventing the averaging out of this form of risk. As a result, market risk cannot be removed by simply putting stocks together in a large portfolio and are *undiversifiable*.

Of the two kinds of risks facing a stock investor, the investor is only compensated for market or systematic risk. Consider the case where investors in companies that faced only idiosyncratic risks were given a risk premium for this diversifiable idiosyncratic risk. An investor could place stocks in a large portfolio and nullify the firm-specific risk. Consequently, this investor would earn a higher return than the risk-free rate for assuming no risk, thereby creating an arbitrage opportunity. However, since an investor cannot eliminate market risk by diversification, an investor's risk premium is determined only by the systematic risk of the investment and not the idiosyncratic risk. Since the return from individual stocks depends only on its market risk, it is not surprising that there is no clear relationship between returns and the volatility (which is the systematic plus idiosyncratic risk). To evaluate the expected return from a stock, a measure of the systematic risk of the firm is needed.

III.2.2 Measuring Systematic Risk – β

It has already been established that an investor's risk premium depends solely on the systematic risk assumed by the investor. However, not all stocks move by the same degree to changes in market fluctuations. While industries like oil and transport move in a similar manner to the entire market, relatively less elastic industries such as tobacco and luxury goods are more immune to market fluctuations. The sensitivity of a stock to systematic risk can be measured by computing the average change in its return caused by a one-percent change in the returns of a portfolio that fluctuates only due to systematic risk. Therefore, to calculate a security's systematic risk, we must first be able to identify such a portfolio that contains only systematic risks and will change in its price only as a response to market-wide economic shocks. A portfolio that cannot be diversified further and contains only systematic risk is called an *efficient* portfolio. In other words, an efficient portfolio is one for which it is impossible to increase the expected returns without increasing the portfolio risk.

We know that diversification increases with the number of stocks included in the portfolio so an obvious candidate for the efficient portfolio is the *market portfolio* itself. The market portfolio contains all stocks and securities that are traded in capital markets. The problem with using the entire market portfolio is that there is not sufficient historical data available for small or newly created companies. As such, the S&P 500 is commonly used as a proxy for the market portfolio. When aiming to compute a

security's market risk, the S&P 500 may be assumed to be fully diversified and is often used as the market portfolio.

We can now measure the systematic risk of the stock as its sensitivity to changes in the returns of the market portfolio. This measure is known as the *beta* (β) of the stock. The β of a stock is defined as the expected percentage change in the return of a stock given a one-percent change in the return of the market portfolio. For example, the β of a utility company such as Consolidated Edison (ED) is 0.28, suggesting a relative immunity to market changes as its stock price changes on average by only 0.28% for a 1% change in the price of the market portfolio. Contrast this with Southwest Airlines (LUV) that has a beta of 1.09, moving 1.09% on average for every 1% change in the price of the market portfolio, almost in-sync with the market. A software company such as Autodesk (ADSK) has a much higher β of 2.14, highly sensitive to market fluctuations.

The eventual goal of determining a stock's risk is to incorporate this risk into a risk premium for the investor's required return. We can now use the β to compute an investor's required return on capital. Since this required return must equal the rate of alternate investments in the market of an equivalent risk, this return is also equal to the equity cost of capital for the stock. To determine this cost of capital, we must first compute the market risk premium. Since the market risk premium is just the additional return expected on investing in the market portfolio above the risk-free rate r_f, the market risk premium is calculated as follows:

$$\text{Market Risk Premium} = E[R_{Mkt}] - r_f \tag{3.13}$$

We know that the β_i for a stock i is just the expected percentage change in the return on investing in i for each one-percent change in the return on the market portfolio. The β_i can therefore be used to determine the degree to which the market risk premium should be adjusted to arrive at the returns for an investment of the same systematic risk level as investing in i. Thus, the cost of capital r_i for investing in stock i is given by:

$$r_i = r_f + \beta_i\ (E[R_{Mkt}] - r_f) \tag{3.14}$$

This is known as the *Capital Asset Pricing Model (CAPM)* and is amongst the most common and important methods of estimating the cost of capital. As a result, it is evident that returns are a function of the risk-free rate and the β of the stock, i.e.,

$$r = \mathrm{f}\left[r_f, \beta\right]$$

It has still not been made clear how the β for a stock is to be computed. We now discuss the mathematical formulas used to compute β and introduce the mathematics required. After developing a means of computing β, it is also important to understand how to compute β with respect to a portfolio of stock investments. The covariance of two returns R_i and R_j is the expected value of the product of their respective deviations from their means and is computed as follows:

$$Cov\left[R_i, R_j\right] = E\left[(R_i - E[R_i])\left(R_j - E\left[R_j\right]\right)\right] \tag{3.15}$$

Similar to the prior instances, historical data is used to estimate the covariance between returns. The method used to estimate the covariance, given historical data from a time period $t = 1 \ldots N$ is given by:

$$Cov\left[R_i, R_j\right] = \frac{1}{N-1}\sum_{t=1}^{N}(R_{i_t} - \bar{R}_i)(R_{j_t} - \bar{R}_j) \tag{3.16}$$

Intuitively, the covariance may be understood as the joint movement of the two stocks. If the two stocks move together, the covariance will be positive and vice-versa. While the sign of the covariance is easily interpreted, its magnitude depends on the returns under observation and is difficult to interpret. We scale down the covariance to a number between -1 and 1 by computing the *correlation* between the stocks. This is more easily interpreted as 1 represents a perfect movement between the two stocks and -1 represents a perfect opposite movement. Independent stocks that have no pattern in their simultaneous movement have a covariance and a correlation of 0.

$$Corr\left[R_i, R_j\right] = \frac{Cov\left[R_i, R_j\right]}{SD[R_i]\, SD\left[R_j\right]} \tag{3.17}$$

Since the β essentially measures the volatility of the stock that is in common with the volatility of the market, the formula to compute the β for a stock i may be computed as:

$$\beta_i = \frac{SD[R_i]\, Corr[R_i, R_{Mkt}]}{SD[R_{Mkt}]} = \frac{Cov[R_i, R_{Mkt}]}{Var[R_{Mkt}]} \tag{3.18}$$

The aim of an investor should be to incorporate stocks into a large portfolio. This helps diversify away the idiosyncratic risks present in the individual stocks. Similar to individual stocks, the β of a portfolio β_P must be calculated to calculate an investor's required rate of return from a portfolio. Since a portfolio is comprised of N number of stocks, the expected return from a portfolio is a weighted average of the returns from the individual stocks. The weight w_i for a stock i is calculated as follows:

$$w_i = \frac{\text{Value of Investment } i}{\text{Total Value of Portfolio}} \tag{3.19}$$

$$R_P = w_1 R_1 + w_2 R_2 + \cdots + w_n R_n = \sum_{i=1}^{n} w_i \, R_i \tag{3.20}$$

$$\beta_P = \frac{Cov[R_P, R_{Mkt}]}{Var[R_{Mkt}]} = \frac{Cov[\sum_{i=1}^{n} w_i R_i \, , R_{Mkt}]}{Var[R_{Mkt}]} = \sum_{i=1}^{n} w_i \frac{Cov[R_i, R_{Mkt}]}{Var[R_{Mkt}]} = \sum_{i=1}^{n} w_i \, \beta_i \tag{3.21}$$

It is clear that the β for a portfolio is just the weighted average of the individual β_i. We can now use the Capital Asset Pricing Model to compute the investor's required rate of the return or the portfolio's equity cost of capital as follows:

$$r_P = r_f + \beta_P (E[R_{Mkt}] - r_f) \tag{3.22}$$

III.2.3. Critique of $\boldsymbol{\beta}$ and the CAPM

Traditionally in financial analysis the β is the quintessential measure of risk and the mean-variance analysis method of the CAPM has been the key tool used to calculate the expected return from a portfolio. However, this paper argues that the β in itself is an insufficient measure of risk and the CAPM is inadequate to fully explain expected returns. The reasons for the insufficiency of β and inaccuracy of the CAPM are highlighted in this section.

The problem with β is that it is essentially a volatility measure as measured through variance or standard deviation. However, it may be the case that an investor

does not think about risk in this manner. Variance is a symmetric dispersion measure, i.e., it measures how far above and below the mean a distribution is spread. Risk in terms of investment is an asymmetric concept, wherein an investor may only be concerned about the probability of loss. Using only the variance will be insufficient in this case, since it doesn't give the investor an indication of what the investor actually thinks about when it comes to risk.

One of the assumptions of the CAPM is that returns are normally distributed, in which case the mean and variance of the returns fully describes the returns distribution. However, some literature suggests that this might not be the case. For instance, between 1916 and 2003 the Dow Jones Industrial Average (DJIA) seemed to have far too many outliers for the data to realistically fit a normal curve. Theoretically, there should have been 58 days when the DJIA moved over 3.4% in either direction, when in reality 1,001 such cases were observed. Such outliers are noticed at swing levels of 4.5% and 7% as well (Mandelbrot, 2004). Analysis using size-sorted portfolio returns in daily, weekly, quarterly and semi-annually intervals shows that in all the mentioned cases, the returns distributions markedly varied from normality (Chung, Johnson, & Schill, 2001). In the absence of a normal distribution, higher order moments may be needed to understand the returns distribution fully and with it the portfolio risk. Aside from this, the CAPM assumes that the β encapsulates all the market risk for a portfolio and that market risk is enough to explain expected returns. As shown in Chapter V, a review of empirical methodology suggests that there are several factors aside from β that can be used to explain a portfolio's returns by more accurately measuring risk.

An issue with using the CAPM is that the model presupposes that the market can be accurately measured by selecting a portfolio that adequately represents the market's composition. However, the S&P 500 for instance, grossly fails to incorporate all of the tradable commodities that are available to the investor. Aside from this, even if measures such as government/corporate bonds, futures, options, derivatives and other financial instruments are accounted for, a vast number of subtler investment opportunities such as art works, jewelry, collections, or even education are not (Roll, 1977). Indeed it is nearly impossible to encapsulate every available investment opportunity, making it nearly impossible to provide a true representation of the 'market'. Note that this does not mean that simply obtaining the best market measure will validate the CAPM. It could be possible that the β measure itself is not the only risk measure that contributes to explaining returns.

Aside from the theoretical weaknesses of the CAPM, or perhaps because of them, the CAPM has been shown to have poor empirical performance (Fama & French, 2004). The final blow to the sole use of β as a measure of risk shall be dealt in the following section, which highlights alternative risk measures and their success in explaining returns.

III.3 Alternative Risk Measures

As mentioned in the previous section, the assumption of normally distributed returns is key to the success of β as a risk measure. If returns follow a normal distribution, then the mean and variance of the distribution must adequately describe

everything an investor needs to know in terms of risk. However, as suggested in the previous section, this assumption is unrealistic and empirical tests seem to reject it (Mandelbrot, 2004). If this is the case, then higher order moments may be needed to describe the distribution, owing to its skewed nature, or the existence of unequal tails. Aside from higher order moments, there are different factors that an investor may think about when it comes to the risks that influence his/her required return. This paper hypothesizes that such forms of alternative risk measures, in addition to the market risk measured by β, are better able to explain observed returns to investors in stock. Some alternative risk measures are now discussed in this section.

III.3.1 Downside Risk/Skewness

Since the CAPM assumes a normal distribution and uses the mean-variance approach to calculate risk, it measures the effect of *all* possible outcomes to gauge the risk of a portfolio. However, risk to an investor is an asymmetric concept, where the investor is concerned more with the possibility of *unfavorable* outcomes. Essentially, it measures what the investor may perceive to be disastrous outcomes. This is the basis for using the *downside risk,* which entails a *mean-semivariance* approach to measuring risk.

Roy (1952) first propounded the use of downside risk measures in what was known as the *safety first* technique. He called the minimum acceptable return to an investor the 'disaster level' and stated that an investor would prefer investments that had the smallest probability of going below this disaster level. Markowitz himself

conceded that investors solely worried about the downside and that the distribution of securities may not be normal (Markowitz, 1959). However, he rejected this approach owing to larger computational requirements and the limited technology available to him at that time. One of his suggestions to measure the downside was using the below-mean or below-target semivariance (SV_M or SV_T). Let N be the number of observations, R_t be the returns at a time period $t = 1 \dots N$, $\bar{R}$ be the mean of the returns and T be the target return for an investor. The SV_M and SV_T can be calculated as follows:

$$SV_M = \frac{1}{N}\sum_{t=1}^{N} max[0, (\bar{R} - R_t)]^2 \tag{3.23}$$

$$SV_T = \frac{1}{N}\sum_{t=1}^{N} max[0, (T - R_t)]^2 \tag{3.24}$$

In fact, the below-mean semivariance gives an indication of how skewed the resulting distribution is (Nawrocki, 1999). Before this is discussed, a quick summation of skewness in the context of a probability distribution may be helpful. Skewness is a measure of the symmetry of a distribution. It measures the third moment of a distribution that is standardized. For a random variable X with mean μ and variance σ^2, and the third central moment μ_3, the skewness γ is computed as follows:

$$\gamma = E\left[\left(\frac{X - \mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\sigma^3} \tag{3.25}$$

To estimate the skewness of a returns distribution using historical data with sample standard deviation s, we use the following formula:

$$\gamma_R = \frac{\frac{1}{N}\sum_{t=1}^{N}(R_t - \bar{R})^3}{\left[\frac{1}{N-1}\sum_{t=1}^{N}(R_t - \bar{R})^2\right]^{\frac{3}{2}}} = \frac{\sum_{t=1}^{N}(R_t - \bar{R})^3}{s^3} \tag{3.26}$$

When a returns distribution is negatively skewed, then the downside returns will have a larger magnitude than the upside returns. This implies that when losses occur, they will tend to be large losses. Figure 3.5 below show the distribution shape for positively and negatively skewed distributions.

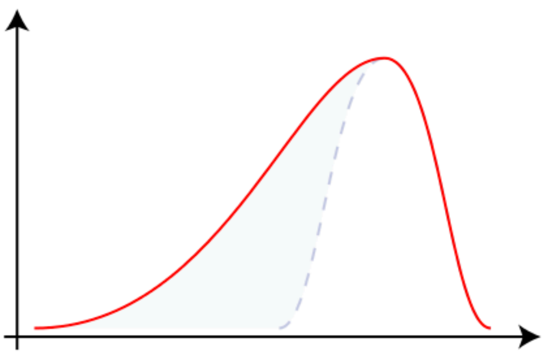

Figure 3.5(a): Positive skewness

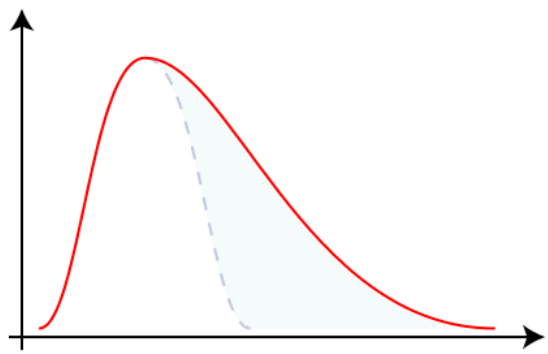

Figure 3.5(b): Negative skewness

A random variable drawn from a normal distribution would have a skewness $\gamma_N = 0$. This is because the distribution is equally spread on either side of the mean. This would imply that the semivariance would be exactly one-half of the variance of the distribution. If one were to take the variance of a normal distribution and divide it by its below-mean semivariance, the ratio would be equal to 2. If the ratio is not 2 then the distribution is skewed or asymmetric. This sets up a theoretical basis for claiming that the semivariance is a proxy measure for the skewness of a distribution.

III.3.2 Tail Risk

While the skewness of a distribution is its standardized third moment and is a measure of the asymmetry of the distribution, the fourth order moment is needed to understand what the probability is of extreme events (specifically extreme losses). In a normal distribution, it is known that approximately 99.7% of the distribution falls between three standard deviations of the mean. As such, the standard deviation gives a sense of how probable *black swan events* are. As the name suggests, black swan events refer to hard-to-predict and extremely rare occurrences. Specifically in relation to finance, it is the occurrence of an event that leads to a huge gain or loss in wealth and lies at the ends of a returns distribution, far from its expected value.

Since a probability density function provides the likelihood of any event occurring, distributions that have *fat tails* are more likely to yield black swan events. A normal distribution has relatively thin tails with very low odds of black swan events occurring. However, as discussed in the prior section, returns distributions may not necessarily be normal. It is crucial to an investor to know how likely it is that the investor may suffer a colossal loss in his/her wealth. The fourth moment of a probability distribution gives an indication of the *peakedness* of the distribution. When the fourth moment is standardized it is called the *kurtosis*. For a random variable X with mean μ and variance σ^2 and the fourth central moment μ_4, the kurtosis κ is computed as:

$$\kappa = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{\mu_4}{\sigma^4} \tag{3.27}$$

To estimate the kurtosis of a returns distribution using historical data with sample standard deviation s, we use the following formula:

$$\kappa_R = \frac{\frac{1}{N}\sum_{t=1}^{N}(R_t - \bar{R})^4}{\left[\frac{1}{N-1}\sum_{t=1}^{N}(R_t - \bar{R})^2\right]^2} = \frac{\sum_{t=1}^{N}(R_t - \bar{R})^4}{s^4} \tag{3.28}$$

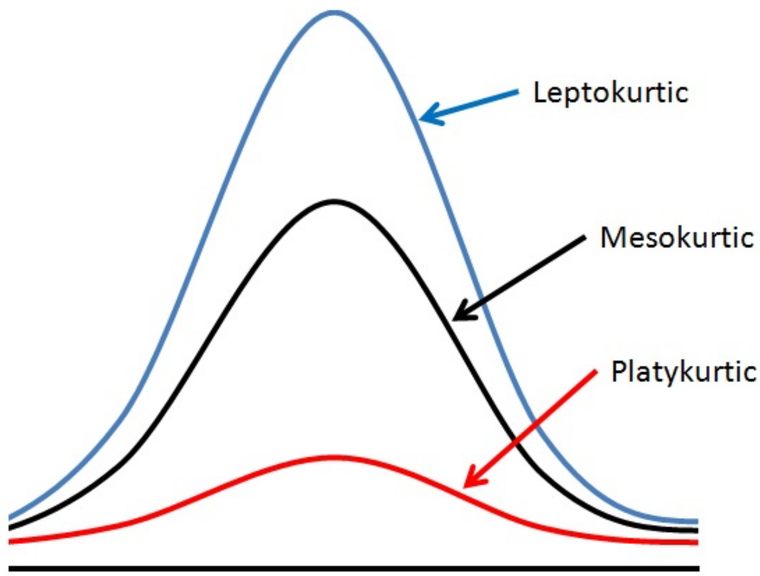

Figure 3.6: Plot showing varying kurtosis for a distribution

A normal distribution has a kurtosis of 3. As such, a measure often used to gauge the fatness of tails is the *excess kurtosis*, or how much kurtosis does a given distribution have over a normal distribution. The excess kurtosis for a normal distribution should obviously equal 0. Distributions with 0 excess kurtosis are called mesokurtic. Distributions that have a positive excess kurtosis are called leptokurtic and have a more acute peak but fatter tails. Negative excess kurtosis distributions are called platykurtic and are flatter with thinner tails. This is shown in Figure 3.6. With specific focus on the financial field, leptokurtic distributions have higher probabilities of black swan event

occurrences. The excess kurtosis κ will be used as a measure of tail risk in the empirical section of this study. The formula to estimate excess kurtosis is given simply by:

$$\kappa = \kappa_R - 3 \tag{3.29}$$

III.3.3 Leverage

The use of leverage by firms increases the risk of investing in that firm because it increases the firm's potential for loss (Horcher, 2005). Leverage is an investment strategy where the firm borrows money to amplify the outcome of a financial transaction. For instance, an automobile manufacturer may borrow funds to construct a factory, with the promise to pay back the funds at a specified future date with interest. The factory will allow the automobile firm to mass-produce cars, which the manufacturer can sell to earn a much higher profit than was possible without the factory. However, leverage does not simply amplify the positive side of a transaction. Consider the case where there is a downturn in the automobile market, and the automobile manufacturer is unable to sell the larger number of manufactured cars owing to the reduction in demand for vehicles. In this case, the manufacturer suffers huge losses and in an extreme case may be unable to pay back the levered amount at maturity. This case would force the automobile manufacturer into bankruptcy.

Several publicly owned companies use leverage as a means to amplify the profits from their transactions. Due to the increase in profits, the firm's earnings increase. This increases the corporation's share price, offering investors a higher return. The amplification is true in the negative scenario as well. If the firm fails to generate enough

revenue to fulfill its increased debt obligations, the firm may be driven to bankruptcy (as was the case with Lehman Brothers in the 2008 financial crisis), leading to massive losses for its shareholders, who are compensated only after the debt holders are paid their money. Owing to this threat, a firm must always plan its cash flows so as to be able to meet its current debt obligations. As such, a firm's leverage should play a role in determining the investor's required rate of return.

A typical measure of the leverage of the firm is its *debt-to-equity ratio (DER)*. It indicates the proportion of equity and debt a company is using to finance its assets ("Debt/Equity Ratio Definition," n.d.). Let D be the total liabilities for a corporation X, and E be its shareholder's equity. The firm's DER is calculated as follows:

$$DER = \frac{D}{E} \tag{3.30}$$

While the DER is the most common measure of a firm's leverage, occasionally measures such as interest-payable long-term debt is used instead of total liabilities to calculate the DER. A high DER suggests that the firm is aggressive in using debt to finance its operations and increase profitability. DER varies significantly between firms and industries. The pertinence to stock returns is that while a high DER increases a firm's ability to make profits, the inability to pay back its debt resulting in bankruptcy is a possibility that could leave an investor with nothing.

III.3.4 The Size of the Firm

A widely observed financial phenomenon is that small firms consistently provide higher returns than larger companies, even when adjusted for market risk (Banz, 1981). It is known in finance as the *small firm effect.* While some investment managers and academics believe this to be an aberration and nothing that an investor may actually take advantage of on a consistent basis (Malkiel, 2003), this paper hypothesizes that it does add explanatory power to stock returns, even after adjusting for the higher market risk that these companies may face. This phenomenon will be tested in this paper using the *market capitalization* (or simply the 'market cap') of a firm as a measure of its size. The market cap is the total dollar market value of all of a firm's outstanding shares ("Market Capitalization Definition," n.d.). It is considered that the firm size is actually a proxy; it is a substitute measure of the underlying risks that are associated with small companies (Crain, 2011). While it is hard to individually identify *all* of the risks a small company faces, it is reasonable to assume that just the size of the firm itself satisfactorily encapsulates these risks.

There are several theories as to why the small firm effect may exist. For instance, investors may demand an additional return on small stock to compensate for their relative lack of information on these companies (Zhang, 2006). This information disparity occurs due to the inevitably less publicity small cap companies receive when compared to their large cap counterparts. Small cap companies receive little to no news

coverage, hampering the rapid dissemination of information, and are focused on less by stock research analysts.

Another possibility for the existence of the small firm effect is the higher *liquidity risk* faced by small firms. Liquidity risks for a firm arise due to its lack of ability to adequately and effectively market its investments. This prevents the investments from being bought and sold fast enough to prevent or minimize a loss. The risk is reflected in widely distorted bid-ask spreads or large/volatile price movements for the firm. Owing to the larger price swings associated with the illiquidity of small cap companies, investors would require a higher equity cost of capital to consider it an attractive investment.

III.4 Concluding Remarks

Chapter III provided an in-depth analysis of modern risk theory. From the theoretical basis of risk, it leads to the variety of measures with which a stock's risk may be determined. It has been shown that a large part of modern financial theory and practice rests on using market risks measured by β as the sole determinant of a stock's risk. Only under this assumption does the CAPM accurately estimate the expected return for a stock. However, as discussed in section III.3, this paper hypothesizes that an approach that utilizes only the β of a stock underestimates risk and leads to inaccurate returns predictions for the stock itself. The argument is, essentially, against the normality of returns distributions. This paper aims to demonstrate, by the use of empirical methods (section VI), that alternate risk measures such as those highlighted in

section III.3 are important in explaining stock returns. If measures other than β are effective in explaining stock returns, then it is plausible that returns are not normally distributed and that market risk is an insufficient tool to determine a stock's expected return.

Note that there may be other alternative risk measures included in prior empirical literature that have not been discussed in this section. Such measures will also be included in the empirical analysis to check for correspondence with prior empirical work and to avoid statistical discrepancies such as omitted variable bias.

In the following Chapter IV, stock prices are analyzed from a mathematical perspective to better understand the nature of a stock price as a random walk. A reader interested only in the effect of alternative risk measures may wish to move directly to Chapter V. Chapter V provides a survey of prior empirical literature where such alternate methods of risk evaluation have been analyzed. The aim is to determine effective empirical methods to test the proposed hypothesis, and identify other key factors such as data sets, empirical methodologies, and statistical tools that other researchers have highlighted when analyzing these measures.

IV. Mathematical Analysis of Stock Prices

In this section the random walk will be explained from a mathematical context. Considering the efficient markets hypothesis to be valid, it has been claimed that stock price changes are a random walk. However, so far the notion of a random walk has been qualitatively described. The random walk can be mathematically formalized as a time series. This is demonstrated in section IV.1. The aim of this section, however, is to show that random walks are a part of a larger system of processes called martingales. Before this can be shown, a deeper understanding of probability measure theory is required including the understanding of σ-algebras and measurability (IV.2), random variables and independence (IV.3), and conditional probability and expectations (IV.4). This is followed by a demonstration of the random walk as a martingale process (IV.5).

IV.1 Taking a Random Walk

To consider price changes as a random walk, let us assume that the price of a stock in corporation Y at time $t = 0$ is P_0. Assume P_0 to be the starting stock market price and consider it as given. Now let $a_1, a_2, \ldots$ be an infinite set of independent and identically distributed (iid) random variables where each a_i has mean 0 and variance σ^2.[3]

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[3] A basic knowledge of mathematical probability theory is assumed from the reader of this thesis. An introduction to the concepts in probability theory may be obtained through some of the references available in the bibliography or from (Leemis, 2011)

If it is assumed that price change from any time period $t = i$ to the next time period $t = i + 1$ is a random walk and that the a_i random variables are interpreted as the change in prices from one time period to the next, then the price at time $t = 1$ is given by:

$$P_1 = P_0 + a_1$$

At time $t = 2$,

$$P_2 = P_1 + a_2 = P_0 + a_1 + a_2$$

At time $t = 3$,

$$P_3 = P_2 + a_3 = P_0 + a_1 + a_2 + a_3$$

So in general, for any time period $t = i$, the price of the stock may be given by:

$$P_i = P_{i-1} + a_i$$

$$P_i = P_0 + a_1 + a_2 + \cdots + a_i \tag{4.1}$$

We use eqn. 3.6 to calculate the expected value of the price of the stock at a time $t = i$.

$$E[P_i] = E[P_0 + a_1 + a_2 + \cdots + a_i]$$

Since P_0 is a constant and each of the a_i's are independent with mean 0,

$$E[P_i] = E[P_0] + E[a_1] + E[a_2] + \cdots + E[a_i]$$

$$= P_0 + 0 + 0 + \cdots + 0$$

$$\therefore E[P_i] = P_0 \tag{4.2}$$

This suggests that the best estimate for the price at any future time period is the known price today ($t = 0$). Since the price changes are independent, there is no way to know the price in the future using historical price data. It is also possible to compute the variance of the stock price at time $t = i$,

$$Var[P_i] = Var[P_0 + a_1 + a_2 + \cdots + a_i]$$

Since P_0 is a constant and each of the a_i's are independent with variance σ^2,

$$Var[P_i] = Var[P_0] + Var[a_1] + Var[a_2] + \cdots + Var[a_i]$$

$$= 0 + \sigma^2 + \sigma^2 + \cdots + \sigma^2$$

$$\therefore Var[P_i] = i\sigma^2 \tag{4.3}$$

Note that the variance is time dependent. It increases linearly with the time period of price estimation suggesting that the scope for error in the prediction of the price at time $t = i$ increases with i. Now consider two time periods i and j such that $1 \leq i \leq j$. We can now compute the autocovariance function for any time periods i, j as follows:

$$\gamma_{i,j} = Cov\left[P_i, P_j\right]$$

$$= Cov\left[P_0 + a_1 + a_2 + \cdots + a_i, P_0 + a_1 + a_2 + \cdots + a_i + a_{i+1} + \cdots + a_j\right]$$

$$= Cov\left[a_1 + a_2 + \cdots + a_i, a_1 + a_2 + \cdots + a_i + a_{i+1} + \cdots + a_j\right]$$

Since the a_i's are all independent, the non-zero values will occur only when taking the covariance of any of the a_i's with itself. Then the autocovariance function,

$$\gamma_{i,j} = Cov[a_1, a_1] + Cov[a_2, a_2] + \cdots + Cov[a_i, a_i]$$

$$= Var[a_1] + Var[a_2] + \cdots + Var[a_i]$$

$$= \sigma^2 + \sigma^2 + \cdots + \sigma^2$$

$$\therefore \gamma_{i,j} = i\sigma^2 \tag{4.4}$$

We know that $Var[P_i] = i\sigma^2$ and $Var\left[P_j\right] = j\sigma^2$. Using this information we can compute the autocorrelation function for P_i and P_j as follows:

$$r_{i,j} = Corr\left[P_i, P_j\right]$$

$$= \frac{Cov[P_i, P_j]}{\sqrt{Var[P_i]}\sqrt{Var[P_j]}}$$

$$= \frac{\gamma_{i,j}}{\sqrt{i\sigma^2}\sqrt{j\sigma^2}}$$

$$= \frac{i\sigma^2}{\sqrt{i}\sqrt{j}\sigma^2}$$

$$\therefore r_{i,j} = \sqrt{\frac{i}{j}} \tag{4.5}$$

In general for any two-time periods i and j the autocovariance and autocorrelation functions may be given by:

$$\gamma_{i,j} = \min[i, j]\,\sigma^2 \tag{4.6}$$

$$r_{i,j} = \sqrt{\frac{\min[i, j]}{\max[i, j]}} \tag{4.7}$$

It is known, however, that stock prices have had a tendency to rise over time, as specified in the introduction of this study and as displayed by the S&P Stock Index displayed in Figure 1.1. While it is true that price changes are a random walk, it may be the case that the progression of the prices may seem to have an upward trend over time. This is known as a *random walk with drift*, and in particular, with a positive drift. To consider stock prices as a random walk with drift, let us again assume that the price of a stock in corporation Y at time $t = 0$ is P_0. Assume P_0 to be the starting stock market price and consider it as given. Now let $a_1, a_2, \ldots$ be an infinite set of independent and identically distributed (iid) random variables where each a_i has positive mean μ and

variance σ^2. We know that in general, for any time period $t = i$, the price of the stock may be given by:

$$P_i = P_{i-1} + a_i$$

$$P_i = P_0 + a_1 + a_2 + \cdots + a_i \tag{4.8}$$

We calculate the expected value of the price of the stock at a time $t = i$:

$$E[P_i] = E[P_0 + a_1 + a_2 + \cdots + a_i]$$

Since P_0 is a constant and each of the a_i's are independent with mean μ,

$$E[P_i] = E[P_0] + E[a_1] + E[a_2] + \cdots + E[a_i]$$

$$= P_0 + \mu + \mu + \cdots + \mu$$

$$\therefore E[P_i] = P_0 + i\mu \tag{4.9}$$

This suggests that the best estimate for the price at any future time period is the known price today ($t = 0$) with an added trend factor that increases with time. Since the price changes are independent, there is no way to know the price in the future using historical price data. It is also possible to compute the variance of the stock price at time $t = i$,

$$Var[P_i] = Var[P_0 + a_1 + a_2 + \cdots + a_i]$$

Since P_0 is a constant and each of the a_i's are independent with variance σ^2,

$$Var[P_i] = Var[P_0] + Var[a_1] + Var[a_2] + \cdots + Var[a_i]$$

$$= 0 + \sigma^2 + \sigma^2 + \cdots + \sigma^2$$

$$\therefore Var[P_i] = i\sigma^2 \tag{4.10}$$

The variance of the prices is the same as the case of the simple random walk and is clearly still time dependent. It may be similarly shown that the autocovariance and

autocorrelation functions are the same as in the case of the simple random walk. The calculation for the autocovariance and autocorrelation is not included in this section.

The discussion on market efficiency has led to the thought that price changes are a random walk and historical data is ineffective in predicting future stock prices. As such, for an investor at any time period t predicting the stock price at time $t + 1$ is a *fair game*. While random walks are a particular example of fair games, the standard probability theory behind fair games is more complex and is modeled by what is known in probability theory as *martingales*. Since predicting stock prices may be modeled using martingales, it is important to discuss the mathematics behind martingale processes. Section IV.2 introduces mathematical concepts that are important in understanding martingales and section IV.3 mathematically describes the martingale process itself.

IV.2 Algebras and Measurability

Measure theory is the mathematics of *measureable sets*. To build the notion of a measureable set, the concept of a *σ-algebra* must first be introduced. Let S be a set. If measure theory is being applied to probabilities, then this set S may be considered to be the *sample space*, which is the set of all possible outcomes of a random process.

Definition 1. A collection Σ_0 of subsets of S is called an *algebra on S* (also as the algebra of subsets of S) if:

i. $S \in \Sigma_0$

ii. $A \in \Sigma_0 \Rightarrow A^c \in \Sigma_0$[4]

---

[4] A^c is the *complement* of set A, i.e. $A^c \coloneqq \{x \in S : x \notin A\}$

iii. $A, B \in \Sigma_0 \Rightarrow A \cup B \in \Sigma_0$

Note that this definition necessarily leads to the following conclusions:

i. $\emptyset \in \Sigma_0 \because \emptyset = S^c$

ii. $A, B \in \Sigma_0 \Rightarrow A \cap B \in \Sigma_0 \because A \cap B = (A^c \cup B^c)^c$ $[By\ De\ Morgan's\ Law]$

Therefore, an algebra on S may be considered as a collection of subsets of S that is stable under a finite number of the set operations of union, intersection, and complement. We now extend the notion of an algebra and define a σ-algebra:

Definition 2. A collection Σ of subsets of S is called a σ-algebra on S (or a σ-algebra of subsets of S) if Σ is an algebra on S such that if $A_n \in \Sigma$ for all $n \in \mathbb{N}$, then

$$\bigcup_n A_n \in \Sigma.$$

Note from the definition that

$$\bigcap_n A_n = \left(\bigcup_n A_n^c\right)^c \in \Sigma \quad [By\ De\ Morgan's\ Law].$$

Thus, the σ-algebra on S is a collection of subsets of S that are stable under any countable collection of the set operations of union, intersection, and complement. It is also clear that for a finite set, any algebra is a σ-algebra. To understand the concept of a σ-algebra better, consider the example presented below:

Example 1. Let the set $S = \{a, b, c\}$. Consider the sets $A = \{\emptyset, S\}, B = \{\emptyset, \{a\}, \{b, c\}, S\}$, and $C = \mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, S\}$.

It is fairly simple work to show that all three sets A, B, and C are σ-algebras on S. It may be noted that the set A represents the *smallest* σ-algebra on S, containing only

two elements, namely the null set and the whole set (or sample space if viewing this from a probability perspective) itself. The set C represents the *power set* of set S. The power set is the set of all subsets of S. Clearly the power set of any set S is a σ-algebra on S. The power set is the *largest* σ-algebra on S. Sometimes the power set of a sample space is too large to be of any use. Now consider the set B. The set B is also a σ-algebra and is smaller than the power set of S. It is also the smallest σ-algebra containing the set $\{a\}$. Therefore, set B is known as the σ-algebra *generated by* $\{a\}$. The method of generating a σ-algebra from a subset of the sample space is fairly easy. To generate a σ-algebra from any subset X of the sample space, simply collect together the set X, its complement X^c, the sample space S, and the null set $\emptyset$. The resulting set is a σ-algebra on S and is also the σ-algebra generated by the set X. The σ-algebra generated by X is the intersection of all σ-algebras on S, which have X as a subset. Likewise if *B* is a collection of subsets of *S*, we can define the σ-algebra generated by *B* to be the smallest σ-algebra on *S* that contains *B*. This σ-algebra exists since it is the intersection of all the σ-algebras on *S* that contain the collection *B*.

In a financial context, the σ-algebra is the set the represents the information that is *known* up until a point in time t. This is better explained with the following illustration: Let $S = \{HH, HT, TH, TT\}$ be the set of all possible outcomes obtained by tossing a fair coin twice, once at time $t = 1$ and again at time $t = 2$. The power set of S is the σ-algebra that represents the situation when everything is known, i.e., at time $t = 2$. However, this result is trivial and now consider the set $\Sigma = \{S, \emptyset, \{HH, HT\}, \{TH, TT\}\}$. The set Σ is a σ-algebra on S and the determination is

left to the reader. It is important to understand that this is the σ-algebra where only the result of the first coin toss at time $t = 1$ is known. It simulates a scenario where time $t = 2$ has not yet occurred. Essentially selecting the set $\{HH, HT\}$ from Σ indicates that a head occurred on the first toss, and the outcome of the second toss is unknown since it could be a head or a tail.

A particularly important σ-algebra is the *Borel σ-algebra* that may be constructed for any topological space. If S is a topological space, then the Borel σ-algebra on S, $\mathcal{B}(S)$ is the family of subsets of S that are generated by open subsets of S.

$$\mathcal{B}(S) \coloneqq \sigma(open\ sets)$$

In particular we define $\mathcal{B} \coloneqq \mathcal{B}(\mathbb{R})$. $\mathcal{B}$ is the most important of all σ-algebras. This is because most of the regularly encountered subsets of $\mathbb{R}$ are elements of $\mathcal{B}$ (Williams, 2008).

Sigma algebras are important in defining *measures*. This shall be discussed further in this section. In measure theory it represents the collection of subsets for which a measure may be defined. In terms of probability theory, and more applicable to the context of financial mathematics such as stock price modeling, it is interpreted as the collection of events to which probabilities may be assigned. σ-algebras are also used in defining conditional expectations as will be discussed further in this chapter. The following definitions and theory must be constructed to arrive at the notion of a *probability measure*.

Definition 3. Let S be a set and Σ be a σ-algebra. The pair (S, Σ) is called a *measurable space* and an element of Σ is called a Σ-measurable subset of S.

Now let S be a set, Σ_0 be an algebra on S, and μ_0 be a non-negative set function such that:

$$\mu_0 : \Sigma_0 \to [0, \infty].$$

Definition 4. μ_0 is *additive* if $\mu_0(\emptyset) = 0$ and for $A, B \in \Sigma_0$ with $A \cap B = \emptyset$,

$$\mu_0(A \cup B) = \mu_0(A) + \mu_0(B).$$

Definition 5. A mapping function μ_0 is *countably additive* or σ-additive if $\mu_0(\emptyset) = 0$ and if $(A_n \in \Sigma_0 : n \in \mathbb{N})$ is a sequence of disjoint sets such that $A = \bigcup A_n \in \Sigma_0$ then,

$$\mu_0(A) = \sum_n \mu_0(A_n).$$

Definition 6. Let (S, Σ) be a measurable space. The map

$$\mu : \Sigma \to [0, \infty]$$

is called a *measure* on (S, Σ) if μ is countably additive. The triple (S, Σ, μ) is called a *measure space*.

If (S, Σ, μ) is a measure space, then this measure space (or even just the measure μ) is *finite* if $\mu(S) < \infty$. The measure is σ-finite if for a sequence $(S_n \in \Sigma : n \in \mathbb{N})$ such that

$$\bigcup S_n = S \; and \; \mu(S_n) < \infty \, \forall \, n \in \mathbb{N}.$$

We now arrive at the definition of a *probability measure* as follows:

Definition 7. A measure μ is called a probability measure and (S, Σ, μ) is called a probability triple if

$$\mu(S) = 1.$$

To differentiate a probability triple from any other measure space, let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability triple where Ω is the sample space with a sigma algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$ such that $\mathbb{P}(\Omega) = 1$. Intuitively, Ω represents the set of outcomes of a probability experiment (such as flipping a coin two times) and ω is an individual outcome of the random process (HT, HH, TT, etc.). The difference between a probability triple and any other measure space is the fact that the probability measure can only lie between 0 and 1.

Definition 8. Let $F \in \mathcal{F}$. F is called $\mathbb{P}$-*null* if $\mathbb{P}(F) = 0$. A statement $\mathcal{S}$ about points ω of Ω is said to hold *almost everywhere (a.e.)* if

$$F := \{\omega: \mathcal{S}(\omega) is\ false\} \in \mathcal{F} \text{ and } F \text{ is a } \mathbb{P}\text{-null set.}$$

Definition 9. A statement $\mathcal{S}$ about outcomes (ω) is true *almost surely (a.s.)*, or *with probability 1 (w.p.1)*, if

$$F := \{\omega: \mathcal{S}(\omega)\ is\ true\} \in \mathcal{F}\ \ and\ \ \mathbb{P}(\mathcal{F}) = 1.$$

IV.3 Random Variables and Independence

We now move on to building the notion of a *random variable* using this probability triple. However, before that is possible, we first revert back to our original notation to build some notation and important definitions. Let (S, Σ) be a measurable space such that Σ is a σ-algebra on S. Let the function $h: S \to \mathbb{R}$. Then $h^{-1}(A) := \{s \in S: h(s) \in A\}$. Recall that $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$.

Definition 10. h is called *Σ–measurable* if $h^{-1} : \mathcal{B} \to \Sigma$, i.e., $h^{-1}(A) \in \Sigma$, for all $A \in \mathcal{B}$.

A Σ - measurable function h may be visualized as follows:

$$S \xrightarrow{h} \mathbb{R}$$

$$\Sigma \xleftarrow{h^{-1}} \mathcal{B}$$

To simplify notation, we use $m\Sigma$ to denote the class of Σ-measurable functions on set S, $(m\Sigma)^+$ to denote the class of non-negative elements in $(m\Sigma)$, and $b\Sigma$ to denote the class of bounded Σ-measurable functions on S. In particular, the *Borel function* is an important outcome. A function h from a topological space S to $\mathbb{R}$ is a Borel function if h is $\mathcal{B}(S)$ measurable. The most obvious and important case is when S is the real numbers $\mathbb{R}$. This relates to our general understanding of a 'function' and is clearly the most commonly used mapping system. We now revert back to talking about a probability measure space and convert the terminology back to the probability triple: $(\Omega, \mathcal{F}, \mathbb{P})$, so that the following may be defined:

Definition 11. Let $(\Omega, \mathcal{F})$ be the family of events (or sample space). A *random variable* X is an element of the class of $\mathcal{F}$-measurable functions on Ω (or $m\mathcal{F}$). Therefore,

$$X: \Omega \to \mathbb{R}$$

$$X^{-1}: \mathcal{B} \to \mathcal{F}$$

A more intuitive understanding of a random variable (henceforth referred to as a r.v.) is that it is a function that maps elements from the probability sample space to corresponding real numbers. However, the above definition is a more mathematically sophisticated method of describing and understanding r.v.'s. The concept of a random variable leads into the following discussion on *independence*. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triple.

Definition 12. Independent σ-algebras: Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be sub-σ-algebras of $\mathcal{F}$. $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are called *independent* if, whenever $F_i \in \mathcal{F}_i$, given $(i \in \mathbb{N})$ and $i_1, i_2, \ldots, i_n$ are distinct, then

$$\mathbb{P}\left(F_{i_1} \cap \ldots \cap F_{i_n}\right) = \prod_{j=1}^{n} \mathbb{P}(F_{i_j})$$

Before we discuss independent random variables, we must first understand what is meant by a σ-algebra generated by a random variable. Let Y be a discrete r.v. such that it takes only the countably many values $y_1, y_2, \ldots$, and let $B_i = \{Y = y_i\}$. Then the B_i will be disjoint sets such that $\bigcup_i B_i = \Omega$. If $\sigma(Y)$ is the collection of all possible unions of the B_i, then $\sigma(Y)$ is a σ-algebra and it is called the σ-algebra generated by Y. It is also clearly the smallest σ-algebra with respect to which Y is measurable.

Definition 13. Independent random variables: Let $X_1, X_2, \ldots$ be random variables. These random variables are said to be *independent* if the σ-algebras generated by them $\sigma(X_1), \sigma(X_2), \ldots$ are independent.

Definition 14. Independent events: The events $E_1, E_2, \ldots$ are said to be *independent* if the σ-algebras $\mathcal{E}_1, \mathcal{E}_2, \ldots$ are independent, where $\mathcal{E}_n$ is the σ-algebra $\{\emptyset, \Omega, E_n, E_n^C\}$.

IV.4 Conditional Probability and Expectations

So far in this section on the analysis of martingales and its relevance to the random walk process, the mathematical discussion on measurability has remained fairly general and widely applicable. To better explain and understand martingales, especially in relation to the random walk process, the following sections will look at a more

condensed version of the material presented above. Firstly, let us only consider the probability measure space (probability triple) $(\Omega, \mathcal{F}, \mathbb{P})$. Here we limit our scope to include only discrete random variables, but these ideas extend naturally to nondiscrete r.v.'s. This allows us to work within the more familiar settings of probability theory that most readers will be familiar with, allowing for an easier transition to martingale theory.

Recall that the conditional probability of event A given event B, $\mathbb{P}(A\,|B)$ is defined as:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

provided $\mathbb{P}(B) \neq 0$. This idea of a conditional probability may be extended to a *conditional expectation*. Some notation must first be developed in order to arrive at a definition for a conditional expectation. Let $A \subset \Omega$. We define the indicator function $1_A(\omega)\colon A \to \mathbb{R}$ as:

$$1_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

Let it be noted that a basic understanding of expectation $\mathbb{E}[X]$ where X is a discrete random variable is assumed of the reader. The notation $\mathbb{E}[X; A]$ means $\mathbb{E}[X \cdot 1_A]$ where 1_A is the indicator function for set A. Another method of writing $\mathbb{E}[X; A]$ is:

$$\mathbb{E}[X; A] = \sum_{X^{-1}\{s\} \cap A \neq \emptyset} s \cdot \mathbb{P}(X^{-1}(\{s\}) \cap A)$$

and

$$\mathbb{E}[X] = \sum_{s \in range(X)} s \cdot \mathbb{P}(X^{-1}(\{s\})) = E[X; \Omega]$$

So far, the discussion on conditional probabilities has resulted in the conception of the conditional probability as simply a number. To encompass a broader understanding of

conditional probabilities, it is better to think of a conditional probability as a random variable. The following is a more precise definition for a conditional probability.

Definition 15. Let $B_1, B_2, \ldots$ be countably many, pairwise disjoint, sets each with positive probability such that $\cup B_i = \Omega$. Let $\mathcal{G}$ be the σ-algebra generated by taking all finite or countable unions of B_i. The conditional probability of a set A given $\mathcal{G}$ is

$$\mathbb{P}(A \mid \mathcal{G}) = \sum_i \frac{\mathbb{P}(A \cap B_i)}{\mathbb{P}(B_i)} 1_{B_i}.$$

This conditional probability definition is easier understood with the assistance of an example. Suppose Ω represents all of the possible outcomes of independently tossing a fair coin three times (i.e. HHH, HHT, TTH, etc.). Consider:

$$\mathcal{F}_1 = \{\emptyset, \Omega, \{HHH, HHT, HTH, HTT\}, \{THH, THT, TTH, TTT\}\}$$

$$\mathcal{F}_2 \text{ contains: } \emptyset, \Omega, \{HHH, HHT\}, \{HTH, HTT\}, \{THH, THT\}, \{TTH, TTT\}$$

Then the set $\mathcal{F}_1$ is the σ-algebra that is analogous to having the first coin toss decided but not the second and third. The set $\mathcal{F}_2$ denotes the events that can be determined after the first two tosses are known. However, the sets listed above are not enough to make $\mathcal{F}_2$ a σ-algebra. To make it a σ-algebra, add to $\mathcal{F}_2$ all the sets that can be obtained by taking unions of these sets. The set $\mathcal{F}_3$ is the set that contains all possible subsets of Ω. This is the situation where all 3 outcomes are known. Let A be the event $\{HHH\}$. We shall compute the conditional probabilities $\mathbb{P}(A \mid \mathcal{F}_1)$ and $\mathbb{P}(A \mid \mathcal{F}_2)$. To compute the $\mathbb{P}(A \mid \mathcal{F}_1)$, let:

$$B_1 = \{HHH, HHT, HTH, HTT\}, \text{ and } B_2 = \{THH, THT, TTH, TTT\}$$

$$\text{Now, } \frac{\mathbb{P}(A \cap B_1)}{\mathbb{P}(B_1)} = \frac{\mathbb{P}(HHH)}{\mathbb{P}(B_1)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4} \text{ and } \frac{\mathbb{P}(A \cap B_2)}{\mathbb{P}(B_2)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(B_2)} = 0$$

$$\therefore\ \mathbb{P}(A \mid \mathcal{F}_1) = (0.25)1_{B_1}$$

The interpretation of this result is that the probability of getting three heads is 0.25 if the first toss is a heads and 0 otherwise. Now we calculate $\mathbb{P}(A \mid \mathcal{F}_2)$. We let

$$C_1 = \{HHH, HHT\}, C_2 = \{HTH, HTT\}, C_3 = \{THH, THT\}, \text{and } C_4 = \{TTH, TTT\}$$

We know that $\mathcal{F}_2$ is the σ-algebra containing all possible unions of these C_i's. Now,

$$\mathbb{P}(A \mid C_1) = \frac{\mathbb{P}(HHH)}{\mathbb{P}(C_1)} = \frac{\frac{1}{8}}{\frac{1}{4}} = 0.5 \text{ and } \mathbb{P}(A \mid C_i) = 0 \text{ for } i = 2,3,4$$

$$\therefore \mathbb{P}(A \mid \mathcal{F}_2) = (0.50)1_{C_1}$$

The plausible interpretation of this statement is that the probability of getting three heads in three flips of a fair coin is 0.50 if the first two flips are heads and 0 otherwise. This example illustrates conditional probabilities but does not illustrate conditional expectations. Let us begin with the definition.

Definition 16. Let $B_1, B_2, \ldots$ be countably many, pairwise disjoint, sets each with positive probability such that $\bigcup B_i = \Omega$. Let $\mathcal{G}$ be the σ-algebra obtained by taking all finite or countable unions of B_i. The conditional expectation of a random variable X given $\mathcal{G}$ is a function on Ω

$$\mathbb{E}(X \mid \mathcal{G}) = \sum_{i=1}^{\infty} \frac{\mathbb{E}[X; B_i]}{\mathbb{P}(B_i)} 1_{B_i}.$$

Note that for notational purposes, we choose to write $\mathbb{E}[X \mid Y]$ instead of $\mathbb{E}[X \mid \sigma(Y)]$.

Proposition 1. $\mathbb{E}[X \mid \mathcal{G}]$ is $\mathcal{G}$-measurable.

Proof. Given a σ-algebra $\mathcal{G}$, recall Definition 10: the definition of a $\mathcal{G}$-measurable function. We know:

$$\mathbb{E}(X \mid \mathcal{G}) = \sum_i \frac{\mathbb{E}[X; B_i]}{\mathbb{P}(B_i)} 1_{B_i}(\omega)$$

We let $S_i = \frac{\mathbb{E}[X;B_i]}{\mathbb{P}(B_i)}$ and let $h: \Omega \to \mathbb{R}$ be the function $\mathbb{E}(X \mid \mathcal{G})$. Since the B_i are all pairwise disjoint, $\omega \in B_i$, for only a particular i.

$$\therefore Range(h) = \{S_1, S_2, S_3, \ldots\} \text{ and } h^{-1}(\{S_i\}) = B_i$$

$$\Rightarrow h^{-1}(A) = \bigcup_{S_i \in A} B_i$$

$$\Rightarrow h^{-1}(A) \in \mathcal{G} \text{ for any } A \subset \mathbb{R}$$

$$\therefore h^{-1}(\text{any } Borel\ set) \in \mathcal{G} \Rightarrow h \text{ is } \mathcal{G} - \text{measurable}$$

Therefore, $\mathbb{E}[X \mid \mathcal{G}]$ is $\mathcal{G}$-measurable. ■

Proposition 2. If $C \in \mathcal{G}$ and $Y = \mathbb{E}[X \mid \mathcal{G}]$, then $\mathbb{E}[Y; C] = \mathbb{E}[X; C]$.

Proof. From definition 16, it is known that:

$$Y = \sum_i \frac{\mathbb{E}[X; B_i]}{\mathbb{P}(B_i)} 1_{B_i} \text{ and } B_i \text{ are disjoint.}$$

$$\Rightarrow \mathbb{E}\big[Y; B_j\big] = \frac{\mathbb{E}[X; B_j]}{\mathbb{P}\big(B_j\big)} \mathbb{E}\, 1_{B_j} = \mathbb{E}\big[X; B_j\big].$$

Since $C \in \mathcal{G}, C = B_{j_1} \cup \ldots \cup B_{j_n} \cup \ldots$, summing the above over j_k gives:

$$\mathbb{E}[Y; C] = \mathbb{E}[X; C].$$ ■

It is important to note that if a discrete r.v. Y is $\mathcal{G}$-measurable, then for any a it must be true that $Y^{-1}(a) \in \mathcal{G}$, which implies that $Y^{-1}(a) \in \mathcal{G}$ is the union of some B_i's.

Since the B_i are all pairwise disjoint, it follows that the discrete r.v. Y must be constant on each B_i.

Proposition 3. Let Y be $\mathcal{G}$-measurable and let $\mathbb{E}[Y; C] = \mathbb{E}[X; C]$ whenever $C \in \mathcal{G}$. Then $Y = \mathbb{E}[X \mid \mathcal{G}]$.

Proof. Given that Y is $\mathcal{G}$-measurable, Y must be constant on each B_i. Let the value of Y on B_i be y_i. $\therefore$ $Y = \sum_i y_i \, 1_{B_i}$. Now,

$$y_i \mathbb{P}(B_i) = \mathbb{E}[Y; B_i] = \mathbb{E}[X; B_i]$$

$$\Rightarrow y_i = \frac{\mathbb{E}[X; B_i]}{\mathbb{P}(B_i)} \Rightarrow Y = \sum_i \frac{\mathbb{E}[X; B_i]}{\mathbb{P}(B_i)} 1_{B_i} = \mathbb{E}[X \mid \mathcal{G}]. \qquad \blacksquare$$

Proposition 4. (1) If $X_1 \geq X_2$, then $\mathbb{E}[X_1 \mid \mathcal{G}] \geq \mathbb{E}[X_2 \mid \mathcal{G}]$.

(2) $\mathbb{E}[aX_1 + bX_2 \mid \mathcal{G}] = a\mathbb{E}[X_1 \mid \mathcal{G}] + b\mathbb{E}[X_2 \mid \mathcal{G}]$.

(3) If X is $\mathcal{G}$-measurable, then $\mathbb{E}[X \mid \mathcal{G}] = X$.

(4) $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]] = \mathbb{E}[X]$.

(5) If X is independent of $\mathcal{G}$, then $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$.

Proof. (1) and (2) immediately follow from the definition. Their proof is straightforward and has been omitted. To prove (3), note that if a random variable $Y = X$, then it is trivial that Y is $\mathcal{G}$-measurable and $\mathbb{E}[X; C] = \mathbb{E}[Y; C]$ for any $C \in \mathcal{G}$. Applying Proposition 3, it follows that $Y = X = \mathbb{E}[X \mid \mathcal{G}]$. To prove (4), let $Y = \mathbb{E}[X \mid \mathcal{G}]$. Now,

$$\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]] = \mathbb{E}[Y] = \mathbb{E}[Y; \Omega] = \mathbb{E}[X; \Omega] = \mathbb{E}[X].$$

Lastly, to prove (5), let $Y = \mathbb{E}[X]$. Y is constant so clearly it must be $\mathcal{G}$-measurable. Given that X is independent of $\mathcal{G}$, if $C \in \mathcal{G}$, then $\mathbb{E}[X; C] = \mathbb{E}[X. 1_C] = (\mathbb{E}[X])(\mathbb{E}[1_C]) =$

$(\mathbb{E}[X])\mathbb{P}(C)$. But we know that $\mathbb{E}[Y;C] = (\mathbb{E}[X])\mathbb{P}(C)$ since Y is constant. $\therefore$ $\mathbb{E}[Y;C] = \mathbb{E}[X;C]$ which implies $\mathbb{E}[X] = Y = \mathbb{E}[X \mid \mathcal{G}]$. ■

Proposition 5. If Y is $\mathcal{G}$-measurable, then $\mathbb{E}[X\,Y \mid \mathcal{G}] = Y\,\mathbb{E}[X \mid \mathcal{G}]$.

Proof. By Proposition 3, we must show that the expectation of $Y\,\mathbb{E}[X \mid \mathcal{G}]$ over sets $C \in \mathcal{G}$ is the same for XY. As was shown in the proof for Proposition 2, consider the case where C is one of the B_i. Since Y is $\mathcal{G}$-measurable, it is constant on B_i. Let the value be y_i. Now,

$$\mathbb{E}[Y\mathbb{E}[X|\mathcal{G}];B_i] = \mathbb{E}[y_i\mathbb{E}[X|\mathcal{G}];B_i] = y_i\mathbb{E}[\mathbb{E}[X|\mathcal{G}];B_i] = y_i\mathbb{E}[X;B_i] = \mathbb{E}[XY;B_i]$$

$$\Rightarrow \mathbb{E}[Y\,\mathbb{E}[X \mid \mathcal{G}];C] = \mathbb{E}[XY;C]$$

$$\Rightarrow \mathbb{E}[XY \mid \mathcal{G}] = Y\,\mathbb{E}[X \mid \mathcal{G}].$$ ■

Proposition 6. If $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$, then $\mathbb{E}[\,\mathbb{E}[X \mid \mathcal{H}] \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{H}] = \mathbb{E}[\,\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}]$.

Proof. $\mathbb{E}[X \mid \mathcal{H}]$ is $\mathcal{H}$-measurable $\Rightarrow$ $\mathbb{E}[X \mid \mathcal{H}]$ is $\mathcal{G}$-measurable, since $\mathcal{H} \subset \mathcal{G}$. The left hand equality is now simply obtained utilizing Proposition 4(3). To obtain the right hand equality, let us first assume that $Y = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}]$. Y is $\mathcal{H}$-measurable, and if $C \in \mathcal{H} \subset \mathcal{G}$, then

$$\mathbb{E}[Y;C] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}];C] = \mathbb{E}[X;C]$$

By Proposition 3,

$$\mathbb{E}[X \mid \mathcal{H}] = Y$$

$$= \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}]$$ ■

Intuitively, Proposition 6 suggests that if we are predicting the outcome of a random variable X given limited information, then this is the same as making a single prediction given the least amount of information. Given the nature of a conditional

expectation as a *predictor* of a random variable given a σ-algebra, it is important to determine whether the conditional expectation is the *best* such predictor. The best predictor is the one with the least mean squared error. If X is the random variable to be predicted, then the predictor Z is just another random variable itself, and the goodness of the prediction will be measured by the expectation $\mathbb{E}[(X-Z)^2]$, or the *mean square error*.

Proposition 7. If X is a r.v. then the best predictor for X among the collection of $\mathcal{G}$-measurable r.v.s is $Y = \mathbb{E}[X \mid \mathcal{G}]$.

Proof. Let Z be any $\mathcal{G}$-measurable random variable. Using Propositions 4(3) and 5, and the fact that Y is $\mathcal{G}$-measurable,

$$\begin{aligned}
\mathbb{E}[(X-Z)^2 \mid \mathcal{G}] &= \mathbb{E}[X^2|\mathcal{G}] - 2\mathbb{E}[XZ|\mathcal{G}] + \mathbb{E}[Z^2|\mathcal{G}] \\
&= \mathbb{E}[X^2|\mathcal{G}] - 2Z\mathbb{E}[X|\mathcal{G}] + Z^2 \\
&= \mathbb{E}[X^2|\mathcal{G}] - 2ZY + Z^2 \\
&= \mathbb{E}[X^2|\mathcal{G}] - Y^2 + (Y-Z)^2 \\
&= \mathbb{E}[X^2|\mathcal{G}] - 2Y\mathbb{E}[X|\mathcal{G}] + Y^2 + (Y-Z)^2 \\
&= \mathbb{E}[(X-Y)^2|\mathcal{G}] + (Y-Z)^2
\end{aligned}$$

Taking expectations on both sides of the equation and using Proposition 4(4),

$$\mathbb{E}[(X-Z)^2] = \mathbb{E}[(X-Y)^2] + \mathbb{E}[(Y-Z)^2]$$

The right hand side of the equation is bigger than or equal to $\mathbb{E}[(X-Y)^2]$ because the term $(Y-Z)^2 \geq 0$. Hence the error in predicting X using Z is bigger than the error when Y is used as a predictor. The two errors will be equal only if $Z = Y$. Therefore Y is the best predictor. ∎

IV.5 Martingales

Consider our previous discussion on stock prices and how a σ-algebra represented the information that was known about the stock from the investment date $t = 0$ till now $t = n$. If measured in discrete time units, this is analogous to a sequence of σ-algebras, $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$, each of one encompasses new price information about the stock at time $t \in \{1,2,\dots,n\}$. Stock prices are but one example of such sequences of σ-algebras. Tossing coins will also produce such a sequence. This section now combines all the previously presented material to arrive at the martingale process.

Definition 17. A r.v. X is said to be *integrable* if the $\mathbb{E}\,|X| < \infty$.

Definition 18. Let $\mathcal{F}_n$ be an increasing sequence of σ-algebras. A sequence of r.v.'s X_n is said to be *adapted* if X_n is $\mathcal{F}_n$ – measurable for each n.

Definition 19. A martingale M_n is a sequence of random variables X_n along with an increasing sequence of σ-algebras $\mathcal{F}_n$ such that:

(1) M_n is integrable for all n,

(2) M_n is adapted to $\mathcal{F}_n$,

(3) and for all n,

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n.$$

Condition (3) is a crucial property. Minor alterations to it allow for a wider branch of processes called submartingales and supermartingales. If we have conditions (1) and (2), and instead of (3) we have:

(a) for all n,

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] \geq M_n$$

then we say M_n is a submartingale, or

(b) for all n,

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] \leq M_n$$

then we say M_n is a supermartingale.

This implies that (contrary to the nomenclature), submartingales tend to increase while supermartingales tend to decrease. Since martingales depend on the collection of σ-algebras, in situations where clarity is required, martingales may also be referred to by $(M_n, \mathcal{F}_n)$.

Having established all the required definitions and propositions necessary to define a martingale process, we may now proceed to the final aim of this section, i.e. to show that the random walk is a martingale.

Proposition 8. The random walk is a martingale process.

Proof. Recall that a random walk is a sequence of random variables P_i where:

$$P_i = P_{i-1} + a_i \text{ and } P_1 = P_0 + a_i$$

given that $a_1, a_2, \ldots$ is a set of iid random variables with mean 0 and variance σ^2 and P_0 is constant. Therefore:

$$P_i = P_0 + a_1 + a_2 + \cdots + a_i$$

We let the sequence of σ-algebras be the constant sequence $\mathcal{F}_n = \mathcal{F}$ where $\mathcal{F}$ is the σ-algebra for the given probability space Ω and probability measure $\mathbb{P}$. We must now show that $\{P_i\}$ is a martingale.

(1) To show the first condition recall that the a_is have finite variance. Therefore,

$$\mathbb{E}[\,|P_i|\,] = \mathbb{E}[\,|P_0 + a_1 + a_2 + \cdots + a_i|\,] \leq \mathbb{E}[\,|P_0| + |a_1| + |a_2| + \cdots + |a_i|\,] < \infty$$

(2) Since we have chosen $\mathcal{F}$ as the σ-algebra for the probability sample space Ω, each a_i is $\mathcal{F}$ - measurable. $P_i = P_0 + a_1 + a_2 + \cdots + a_i$ is also $\mathcal{F}$-measurable since P_0 is a constant and the sum of the $\mathcal{F}$-measurable random variables a_i is $\mathcal{F}$-measurable (Williams, 2008). Therefore, P_i is $\mathcal{F}_n$ adapted.

(3) We know,

$$\mathbb{E}[\, P_i \mid \mathcal{F}\,] = \mathbb{E}[\, P_0 + a_1 + a_2 + \cdots + a_i \mid \mathcal{F}\,]$$

$$= P_0 + \mathbb{E}[\, a_1 \mid \mathcal{F}\,] + \mathbb{E}[\, a_2 \mid \mathcal{F}\,] + \cdots + \mathbb{E}[\, a_i \mid \mathcal{F}\,] = P_0$$

Now look at,

$$\mathbb{E}[\, P_{i+1} \mid \mathcal{F}\,] = \mathbb{E}[\, P_i + a_{i+1} \mid \mathcal{F}\,] = \mathbb{E}[\, P_i \mid \mathcal{F}\,] + \mathbb{E}[\, a_{i+1} \mid \mathcal{F}\,] = P_0 + 0 = P_0 = P_i$$

Therefore, it has been shown that $\{P_i\}$ is a martingale. ■

Proposition 9. The random walk with positive drift is a submartingale.

Proof. Recall that a random walk with drift is a sequence of random variables P_i where:

$$P_i = P_{i-1} + a_i \text{ and } P_1 = P_0 + a_i$$

given that $a_1, a_2, \ldots$ is a set of iid random variables with mean $\mu > 0$ and variance σ^2 and P_0 is constant. The first two conditions are proved similar to (1) and (2) in the previous proof. Only condition (3) must be shown:

(3) We know,

$$\mathbb{E}[\, P_i \mid \mathcal{F}\,] = \mathbb{E}[\, P_0 + a_1 + a_2 + \cdots + a_i \mid \mathcal{F}\,]$$

$$= P_0 + \mathbb{E}[\, a_1 \mid \mathcal{F}\,] + \mathbb{E}[\, a_2 \mid \mathcal{F}\,] + \cdots + \mathbb{E}[\, a_i \mid \mathcal{F}\,] = P_0 + i\,\mu$$

Now look at,

$$\mathbb{E}[\, P_{i+1} \mid \mathcal{F}\,] = \mathbb{E}[\, P_i + a_{i+1} \mid \mathcal{F}\,] = \mathbb{E}[\, P_i \mid \mathcal{F}\,] + \mathbb{E}[\, a_{i+1} \mid \mathcal{F}\,] = P_0 + (i+1)\mu > P_i$$

Therefore, it has been shown that $\{P_i\}$ is a submartingale. ■

This concludes the discussion on random walk theory and martingales. It has been shown that if the efficient markets hypothesis holds, then stock prices are a random walk, which means that they are martingale processes. As such, a test for stock price normality may be to test for the properties of a martingale. If any of the properties held by martingale processes do not hold for the stock price data, then this suggests that stock prices may not be normally distributed.

V. Review of Empirical Literature

In Chapters II and III, the theoretical basis for using alternative risk measures was established. It was shown that using the β of a stock or a portfolio may be insufficient in situations where returns are not normally distributed. Prior empirical literature suggested that returns weren't necessarily normal, often deviating from the shape of a normal curve. In section V of this study, we look at some of the prior empirical research conducted on the performance of alternative risk measures and their ability to explain stock returns. This section serves as a link between the theoretical foundations established in prior sections and the empirical work conducted in the next chapter. This is because these prior empirical studies provide key information regarding empirical techniques and data sources that will be necessary to establish a valid and reliable empirical methodology to test this paper's hypothesis.

Eugene Fama and Kenneth French's empirical research is, perhaps, the pioneering work in this field even if it may not be the first. It paved the way for future research in alternative risk measures. Owing to this, it is the first empirical article to be discussed in this chapter (V.1). In particular, they find that the firm's β, book-to-market ratio, and size are important in predicting its returns. Their model is known commonly as the *Fama-French Three-Factor Model*.

The Fama-French study utilizes a cross-sectional regression methodology developed in a study conducted by Fama and Macbeth. This study is discussed in section

V.2. The Fama-Macbeth study tests for the validity of the CAPM but it is the methodology developed by authors that is more important to this thesis. Another study conducted by Black, Scholes and Jensen presents a portfolio methodology that is different from the cross sectional regression method developed by Fama-Macbeth. Since this thesis aims to test the hypothesis using a variety of testing methods, this empirical article is discussed in section V.3.

This paper hypothesizes the importance of higher order moments such as skew and kurtosis, as well as measures such as downside risk and leverage. As such, empirical work such as Chung, Johnson and Schill (2006), Estrada (2007), and Bhandari (1988) is reviewed as these studies have previously tested the variables hypothesized by this study to check if they are important in explaining returns. For each of the articles reviewed in this chapter, the model constructed by the authors is described, followed by a discussion on the data used and the methodology of testing, and finally a discussion of their results and limitations and the relation of the article to this study.

V.1 Fama and French

Fama and French (Fama & French, 1992) noted several empirical contradictions to the CAPM. Empirical studies had already found a relationship between a firm's size, leverage, book-to-market ratio (BE/ME), and earnings-to-price ratio (E/P) and its average returns. However, these studies were univariate and Fama-French aimed to conduct a multivariate analysis of expected returns on all of these variables

simultaneously, to see the effect these variables have on average returns for the sample time period from 1963 to 1990.

V.1.1 The Model

Fama and French aimed to create a multivariate model to explain expected returns. The conceptual variables and their hypothesized relationships are presented in equation 5.1 below:

$$\text{Expected Returns} = f[Volatility^{+}, Size^{-}, Leverage^{-}, Firm\,Value^{+}, Equity\,Value^{+}] \tag{5.1}$$

The authors operationalize the conceptual hypothesis by using the following variables respectively:

$$\text{Expected Returns} = f\left[\beta^{+}, Market\,Equity^{-}, Debt^{-}, \left(\frac{BE}{ME}\right)^{+}, \left(\frac{E}{P}\right)^{+}\right] \tag{5.2}$$

V.1.2 Data and Methodology

Fama-French used panel data to see if there was enough evidence to support their hypotheses. The data was cross sectional in that it comprised of monthly average returns on the NYSE, AMEX, and the NASDAQ. The sample time range is from 1963 to 1990. All non-financial firms available through the NYSE, AMEX, or NASDAQ were selected if they intersected with the merged COMPUSTAT annual industrial files, which contained income statements and balance sheets. Both the returns data and the COMPUSTAT data were maintained by the Center for Research in Security Prices (CRSP).

Financial firms were excluded from selection owing to the large amount of leverage that is commonly assumed by such companies. In general, it is more likely that leverage is an indicator of distress in non-financial firms. The COMPUSTAT data provides reliable data only from 1962 as prior to this date, not only is the data generally unavailable, but also exhibits a strong survivorship bias towards large historically successful companies.

To ensure that future data is not being used to predict past returns, it is essential that accounting variables be offset by a time lag so as to ensure that they are *known* prior to the realized returns. As such, accounting variables from a time period $t-1$ (starting from 1962) are used to predict the returns in time period t (beginning in 1963). The conservative one-year time lag ensures that non-compliance with SEC filing regulations are overcome. Fama-French use market equity data from December in year $t-1$ to compute the book-to-market, leverage, and earnings-to-price ratios at time $t-1$. Market equity from June at time t is used to compute firm size. Fama-French reports that using fiscal year end market equity does not significantly affect their results.

For the testing of the hypothesis, the cross-sectional regression methods used in Fama & MacBeth (1973) (referred to as FM asset-pricing tests) are utilized. However, the β for the stocks must first be calculated. Fama-French create portfolios to compute the βs since estimates from portfolios are more precise than estimating βs for individual stocks. The β for each portfolio is calculated and then assigned to each of the stocks in that portfolio. Size, E/P, leverage, and BE/ME are measured accurately for the individual stocks. As such, the creation of portfolios to calculate these variables is unnecessary.

To estimate the βs for the stocks, all the NYSE stocks are sorted by size (i.e. ME) in the June of each year. This determines the decile breakpoints for the stocks and creates 10 size portfolios. Only the NYSE is used to create size deciles so as to avoid creating portfolios that would contain many small stocks owing to the nature of the NASDAQ added after 1973. Portfolios are created based on size since the work of CITE suggests that size produces a wide range of average returns and β values. The problem with this approach, however, is that the size and β values are highly correlated ($r = -0.988$). This reduces the power of asset-pricing tests to distinguish the effects of size from β in explaining average returns. To allow for this separation, each size decile is then further subdivided by pre-ranking βs for individual stocks. These pre-ranking βs are calculated by using monthly returns in the 5 years preceding July of the current year t. For the same reason mentioned above, only NYSE stocks are used to conduct the pre-ranking. After the assignment of firms is complete, there are now 100 portfolios that are available, separated on size and pre-ranking βs.

After this preliminary work is concluded, the equally weighted monthly returns on the portfolios are calculated for the next 12 months (July to June). As a result, post-ranking monthly returns are available from July 1963 to December 1990 (330 data points). βs are computed on the full sample size and for each portfolio. The β value for each portfolio is computed as the sum of the slope coefficients of the regression of returns on a portfolio on the current and prior month's market (CRSP value-weighted portfolio) return. Correcting for autocorrelation biases does not significantly affect the β values so the simple sum approach is used. The β values so calculated are assigned to

each stock in the size $-\beta$ sorted portfolio and is then used in the Fama-Macbeth regressions to evaluate the effect on returns. Fama-French claim that the precision obtained from this β calculation method outweighs the fact that the true βs are not the same for each stock in a portfolio.

V.1.3 Results and Conclusion

After conducting regression analysis on the data collected, the authors present results that are available from Tables 1, 2, and 3[5]. The tables provide numerical evidence for the results that are mentioned subsequently. When the β values are tested on a stand-alone basis, the paper shows (look at t-scores from Table 3) that β does not have strong explanatory power for returns. This result recurs when β is used in conjunction with size, with size significantly overshadowing the explanatory power of β. Additional tests suggest that the effect of leverage (A/ME and A/BE) is captured by the book-to-market ratio. Also, while the E/P ratio is statistically significant from a univariate perspective, in the presence of both ME and BE/ME, E/P is insignificant, suggesting that ME and BE/ME captures the effect of E/P.

While the main result of this study is the apparent contradiction to the CAPM and its empirically supported claim that β is the sole risk element that affects security returns, in terms of its importance to this overall study, the effect of the other variables is more crucial. It is clear from the results that size and book-to-market ratio are highly significant in explaining returns variation. The former seems to act as a proxy for the

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[5] All tables for literature review results are available in the Appendix

P/E ratio while the latter acts as a proxy for leverage. The results also hold true for the NASDAQ and AMEX markets for the 1963 to 1990 time frame.

A troubling element of the Fama-French study is the lack of a theoretical section that explains why economic and financial theory would suggest that the tested variables be of investigative value. This weakness has already been addressed by the current study in Chapters II and III.

The Fama-French study provides a suitable empirical methodology and framework that is suitable to testing the hypothesis of this paper, adding measures for skewness and kurtosis. Owing to the conduction of this study in 2015, a variety of time samples will also be used and not the 1963 to 1990 time frame of Fama and French. It will be insightful to learn how the results from this study are affected during turbulent economic times, such as the recently concluded 2008 financial crisis.

V.2 Fama and Macbeth

In this study, Fama-Macbeth (Fama & MacBeth, 1973) test for a relationship between the average returns and risk for NYSE common stocks. In particular, the authors aim to test the “two-parameter” portfolio model. The two-parameter model is simply the Capital Asset Pricing Model that has been discussed in detail in Chapter III. This study is interested in both the methodology used to test the two-parameter model rather and the actual results themselves. This is because the Fama-Macbeth regression technique used in this paper is used as the standard testing tool in the work of Fama & French, 1992 and Jensen et al., (2006) which will be discussed in the next section). It is

also the testing methodology that will be used in this study. Additionally, the testing of the CAPM efficacy in explaining returns is important as this study claims that alternative risk measures *adds* explanatory power to the CAPM.

V.2.1 The Model

The theoretical foundations for their model are first laid out by Fama-Macbeth. The theoretical basics for the CAPM established by them are similar to those discussed in this study in Chapter III. Based on the theory, the authors have three testable implications of the CAPM: (1) The expected return to risk relationship for an efficient portfolio is linear, (2) β is a complete measure of the risk of a security that is part of an efficient portfolio, and (3) there is a positive risk-return relationship. While the CAPM is expressed in terms of expected returns, testing it requires real data on period-by-period security returns. The authors use the following stochastic generalization:

$$\widetilde{R_{it}} = \widetilde{C_{0t}} + \widetilde{C_{1t}}\,\beta_i + \widetilde{C_{2t}}\,\beta_i^2 + \widetilde{C_{3t}}\,s_i + \epsilon_{it} \tag{5.3}$$

where $\widetilde{R_{it}}$ is the one-period percentage return on a security i from time period $t-1$ to t. The variable β_i^2 is used to test for linearity. If condition (1) is true, then $E\left[\widetilde{C_{2t}}\right] = 0$. The variable s_i is some measure of risk for security i that is deterministically different from β_i. Condition (2) specified above suggests that the $E\left[\widetilde{C_{3t}}\right] = 0$. Condition (3) is tested by testing the hypothesis that $E\left[\widetilde{C_{1t}}\right] > 0$. This model tests all three of these hypotheses. The statistical disturbances ϵ_{it} are assumed to be independent of the other variables and have mean 0. Also, since this is a test of the CAPM, which allows for riskless borrowing and lending, there is a fourth hypothesis: $E\left[\widetilde{C_{0t}}\right] = r_{ft}$, the riskless

rate of borrowing. Additionally, market efficiency (an assumption made by the CAPM) requires that all the stochastic coefficients and the statistical disturbance be fair games.

V.2.2 Data and Methodology

Similar to the Fama-French model discussed previously, Fama-Macbeth use the CRSP data set to gather monthly percentage returns for all the common stocks traded on the NYSE from January 1926 through to June 1968. It is important to note that when computing the monthly returns from historical prices, the appropriate adjustments are made to account for dividend payments and capital changes such as splits or stock dividends. This is something to bear in mind during the data collection phase of this study as well. Fisher's Arithmetic Index is the proxy chosen to represent the market. It is an equally weighted average of all stocks listed in the NYSE in a given time period (Fisher, 1966).

To begin the analysis, the appropriate values for the independent variables in equation 5.3 must be computed. Since this is an empirical test, the values must be computed through the use of estimators. The following equation states the means of calculating $\widehat{\beta_i}$, the estimator for β_i.

$$\widehat{\beta_i} = \frac{\widehat{Cov}\left[\widetilde{R_i}, \widetilde{R_{Mkt}}\right]}{\widehat{Var}\left[\widetilde{R_{Mkt}}\right]} \tag{5.4}$$

where $\widehat{Cov}\left[\widetilde{R_i}, \widetilde{R_{Mkt}}\right]$ and $\widehat{Var}\left[\widetilde{R_{Mkt}}\right]$ are the estimators used to estimate the covariance between the security and the market and the market variance respectively. Note that the formulas used to estimate the various moments based (skewness and

kurtosis) have already been discussed in section III.3. The consistency in methodology is nonetheless reassuring. However, as is the case with Fama-French, Fama-Macbeth do not compute βs for individual securities. They create portfolios based on ranked βs and use the portfolios in the regression analysis.

To compute the non-β risk of the securities, $s(\widehat{\varepsilon_i})$ is used. The variable $s(\widehat{\varepsilon_i})$ is the standard deviation of the OLS residuals $\widehat{\varepsilon_i}$ from the *market model* presented in equation 5.5 below:

$$\widetilde{R_{it}} = \alpha_i + \beta_i \, \widetilde{R_{Mkt}} + \widetilde{\varepsilon_{it}} \tag{5.5}$$

This standard deviation measure $s(\widehat{\varepsilon_i})$ is a non-β risk measure for a security as it is a measure of total dispersion of the security's returns distribution. The risk measures are calculated from monthly returns from 1930 to 1935, 1936, or 1937. Twenty portfolios are created, with equal weighting of individual securities for each month. The month-by-month returns on these portfolios are also computed for the 4-year period from 1935 to 1938. These returns data are used as the dependent variable and the risk variables are used as independent variables, for each month t of this 4-year period in the following cross-sectional regression:

$$R_{p,t} = \widehat{C_{0t}} + \widehat{C_{1t}} \, \beta_{p,t-1} + \widehat{C_{2t}} \, \beta^2_{p,t-1} + \widehat{C_{3t}} \, \bar{s}_{p,t-1}(\widehat{\epsilon_i}) + \varepsilon_{p,t} \tag{5.6}$$

$$p = 1, 2, \dots, 20$$

where the independent variable $\beta_{p,t-1}$ is the average of the β_i for all the securities in the portfolio p, $\beta^2_{p,t-1}$ is the average of the squared values of the β_is, and $\bar{s}_{p,t-1}(\widehat{\epsilon_i})$ is the average of the $s(\widehat{\epsilon_i})$ for all the securities in portfolio p. To consult the summary statistics of the testing data, turn to Table 4 in the Appendix.

V.2.3 Results and Conclusion

Following regression analysis, Fama and Macbeth tabulate their results as demonstrated in Table 5. A key result is that the alternative risk measure $s(\hat{\epsilon})$ is statistically insignificant. This result suggest that in the regression analysis conducted by Fama-Macbeth, β is the best explanatory risk variable among those selected for analysis. The lack of statistical significance of the squared power of β provides evidence for the linearity of the relationship between returns and risk (using β as the measure for risk). The significance of β, however, does not undermine the hypotheses proposed by this paper, as the authors Fama-Macbeth do not test the significance of β under the presence of the alternative risk measures proposed by this study. However, the linear dependence is a beneficial result, as it provides empirical evidence for using a multivariate regression model in the empirical analysis segment for this study.

It is important to note that the portfolio method used by Fama-Macbeth will not be used in this study. This is because the exact values for variables such as debt/equity, book-to-value, size, etc. are known. In conclusion, the Fama-Macbeth paper provides useful information on using the cross-sectional regression approach to evaluating alternative risk measures. It also provides the empirical background for using a linear regression model. In addition, it provides some groundwork in analyzing the utility of β as a risk measure.

V.3 Black, Scholes, and Jensen

In this study (Jensen et al., 2006), researchers Black, Scholes, and Jensen conduct empirical tests to check the validity of the CAPM. While the aim of the paper is similar to the Fama-Macbeth study discussed in the previous section, the Black, Scholes and Jensen is different and also important, owing to its varying testing methodology. This study utilizes a portfolio construction method that is different from the cross-sectional regression methodology of Fama-Macbeth and Fama-French. It is an important piece of empirical literature in relation to this thesis, as a simplified version of the portfolio methodology utilized by Black, Scholes and Jensen will also be utilized to test this study's proposed hypotheses. We will begin by a brief summary of the model constructed by Black, Scholes, and Jensen and then discuss the data and methodology utilized by the authors. This section concludes with a discussion of results and the link between this study and that of Black, Scholes, and Jensen.

V.3.1 The Model

The hypothesis of Black, Scholes and Jensen's study is that security returns are not strictly proportional to β. Given the hypothesis that the β is insufficient to explain returns, the authors also suggest the need for an additional variable that captures the risk that is not accounted for by β, a claim that this paper also tests. The CAPM is stated in terms of ex-ante returns, which are unobservable. In order to test the hypothesis, Black, Scholes and Jensen utilize ex-post returns. Given the prediction made by the

CAPM, the difference between expected and actual excess returns α_i should equal 0, i.e., if the CAPM is valid,

$$\alpha_i = E(R_i) - r_f - \beta_i\big(E(R_{Mkt}) - r_f\big) = 0 \tag{5.7}$$

The Black, Scholes and Jensen study tests whether α_i is statistically different from 0. Therefore, the following hypothesis is tested empirically:

$$H_0: \alpha_i \neq 0, \qquad H_A: \alpha_i = 0$$

V.3.2 Data and Methodology

The data used for hypothesis testing is a panel data set that contains stock data from the stocks listed on the NYSE during the time period from 1926 to 1966. The data are obtained using the CRSP data set, similar to the Fama-French and Fama-Macbeth studies. The yield on 30-day US T-Bills for the time period 1948 to 1966 is chosen as the risk-free rate. For the time period 1926-47, return on dealer commercial paper is chosen as the risk-free rate since T-Bills were unavailable during that time. While the data collection methodology is fairly standard, the methodology used for testing the Black, Scholes and Jensen hypothesis was new and innovative as it used portfolios to test for statistical significance instead of individual securities. The advantage of using portfolios is that it allows for diversification, eliminating risk for which the investor is not compensated. This allows for a better method of isolating the dependency of returns on market risk. The portfolio method is also more efficient than the individual stocks method as data are available for a large number of individual stocks.

To test for statistical dependence, the portfolios constructed are ranked on the basis of β values. A two-step procedure is utilized to prevent selection bias that would arise from using the estimated β values for each security for both ranking and regression analysis. First, the data from the previous five years is used to estimate βs for each stock. The individual stocks are then ranked in descending order and grouped to form 10 portfolios. Following this, monthly returns for each portfolio are calculated for the following 12 months. At the end of this 12 month period, the 5-year period is updated to the immediately preceding 5-years at the ranking methodology is repeated. This suggests that a security in one portfolio during one time period may be in a different portfolio in a subsequent time period. This process is repeated for the 35-year testing time period. At the end, the average returns α_is is calculated to test for statistical significance using t-tests. Summary statistics for portfolio returns are available in Table 6. A correlation test between the estimated returns and the actual market returns demonstrate a strong positive correlation as expected. In addition, the model is tested for autocorrelation of error terms between successive time periods. The null hypothesis that there is no correlation cannot be rejected.

V.3.3 Results and Conclusion

The main result obtained from the Black, Scholes and Jensen study is that stock returns are not proportional to the β of the stock in the manner suggested by the CAPM (observe the coefficient values for returns and β in Table 7). In terms of the hypothesis tested, there is not enough statistical evidence to reject the null hypothesis that $\alpha_i \neq 0$.

The authors discover that contrary to the CAPM, low β securities seemed to perform *better* than high β securities, i.e. for low β securities, $\alpha_i > 0$ on average and for high β securities, on average $\alpha_i < 0$. Owing to this surprising result, the authors construct a new two-factor model that includes a second factor called the 'β factor', r_z. This factor is defined in a later study by Black as the return on a portfolio that has zero covariance with the return on the market portfolio. As such, the two-factor model to be tested is as follows:

$$E(r_i) = E(r_z)(1 - \beta_i) + E(r_{Mkt})\beta_i \tag{5.8}$$

The beta factor utilized above is shown to have a value between 1.0% and 1.3%, which is statistically different from the risk-free rate. The authors also conduct a cross sectional analysis to determine the significance of the two-factor model. It is important to note that Black, Scholes and Jensen do not conduct heteroskedasticity tests. Given the panel data set utilized by the study, there is a likelihood that the results obtained are subject to heteroskedasticity. As such, the t-scores and statistical conclusions of the study should be viewed with less certainty as they may be skewed due to the presence of heteroskedasticity.

The Black, Scholes and Jensen study is an important contribution to literature in general and to this study in particular. A simplified version of the portfolio method described above is used for hypothesis testing by this study. Instead of using t-score testing, a simple rank correlation method is utilized to test for statistical significance. Aside, from β, alternative risk measures are also used to rank portfolios. The methodology is further described in the next chapter.

V.4 Chung, Johnson and Schill

This study (Chung et al., 2001) analyzes the effect of higher order co-moments on the returns generated from stocks. As discussed in prior sections, empirical evidence has suggested that stock returns may not be normally distributed. This serves as the motivation for the Chung, Johnson and Schill study. Since the non-normality of returns implies that the CAPM is unlikely to hold, the authors claim that there is a need to include higher order moments so as to adequately describe the shape of the returns distribution. The authors suggest that the non-market risk factors used in the Fama-French study such as size (SMB) and book-to-market ratio (HML) are simply proxies for risk factors that can be measured using higher order moments.

The Chung, Johnson and Schill study is an important contribution to literature as it tests the ability of higher order moments (up to the 10th degree) to predict stock returns. It is particularly important with respect to this study as this study hypothesizes skewness and kurtosis as alternative risk measures. Skew and kurtosis represent third and fourth degree moments. Note that there is a theoretical explanation behind using skew and kurtosis while theoretical reasoning behind moments of order higher than 4 becomes difficult to assess. The authors summarize the effect of higher orders as providing information on the likelihood of obtaining extreme outcomes. However, the exact nature of their effect is hard to convert into an intuitive theoretical understanding of what element of risk is being measured.

V.4.1 The Model

The authors examine whether daily, weekly, monthly, quarterly, or semi-annual returns exhibit a normal distribution. Also recall that the authors' aim is to show that in the presence of higher order moments, the Fama-French three-factor model is insignificant. The model tested by Chung, Johnson and Schill is given by:

$$r_{i,t} = a_1 + a_{SMB}s_{i,t} + a_{HML}h_{i,t} + \sum_{j=2}^{n} a_j b_{i,j,t} + e_{i,t} \tag{5.9}$$

where $s_{i,t}$ and $h_{i,t}$ are the factor loading for SMB and HML respectively and the $b_{i,j,t}$ is the i^{th} systematic co-moment. Given equation 5.9 above, the following hypotheses are to be tested:

$$H_0: a_{SMB} \neq 0, a_{HML} \neq 0, a_j = 0 \text{ for all } j$$

$$H_A: a_{SMB} = 0, a_{HML} = 0, a_j \neq 0 \text{ for some } j$$

V.4.2 Data and Methodology

Similar to the prior studies discussed in this literature review, the authors use the CRSP database to collect stock returns data on a daily, weekly, monthly, and semi-annual basis for the time period from 1930 to 1998. The COMPUSTAT database is used to collect data on SMB and HML for the time period from 1970 to 1998. For the risk-free rate, the 30-day T-Bill rate is used. To compute the market index, CRSP data covering the NYSE and AMEX until 1993 is used. Post 1993, NASDAQ data is added to the market index calculation.

To test the model presented in equation 5.9, the authors utilize the Fama-Macbeth cross-sectional regression methodology presented in section V.2. However, instead of ranking the portfolios on the basis of the β values, Chung, Johnson and Schill rank all the stocks included in the CRSP dataset on the basis of market cap (size) to form 50 portfolios of equal size. Time series analysis is used to compute returns for each portfolio for the specified time period and the T-Bill rate is subtracted to obtain the excess returns. The same procedure is repeated to create 50 portfolios ranked by HML. These excess returns are used to estimate the regression equation 5.9 above to test for the significance of SMB and HML in the presence of higher order co-moments.

V.4.3 Results and Conclusion

Results from the Chung, Johnson and Schill study are found in Tables 9 and 10 in the Appendix. The most important result of the Chung, Johnson and Schill study is that in the presence of higher order co-moments, the t-scores of SMB and HML almost always drop drastically. The addition of higher order co-moments also causes adjusted-R^2 values to increase, suggesting that each additional co-moment increases the model's explanatory power.

In addition to testing for the significance of Fama-French model in the presence of higher order moments, the authors also check whether the impact of higher order moments varies with respect to the time period of observation: daily, weekly, monthly, quarterly, or semi-annually. The results suggest that the effect of higher order co-moments increases proportionally with the time horizon under consideration. In

particular, from monthly returns onwards, the normality of returns is rejected for all cases and skew and kurtosis become increasingly important. For this reason, this thesis uses monthly returns as it allows for a large data sample and does not display normal returns distributions. It should be noted here that the testing methodology of Chung, Johnson and Schill does not incorporate the testing of variables such as β and downside β, which may be important in explaining returns. Owing to this, the results may be subject to an omitted variable bias.

The Chung, Johnson and Schill study is important as it provides evidence on the empirical benefits of using higher order co-moments as alternative risk measures. Owing to the lack of a concrete theoretical basis for using moments of higher order than 4, this paper limits the use of higher order moments to skewness and (excess) kurtosis. Also, this study utilizes monthly returns data, owing to the larger availability of data and lower likelihood of obtaining of normal returns distributions.

V.5 Javier Estrada

Prior to this section, we have reviewed empirical literature with respect to articles that contain important empirical methodologies (Fama & MacBeth, 1973, Jensen et al., 2006) and/or investigate relevant alternative risk measures such as β, size and price-to-book (Fama & French, 1992), and skew and kurtosis (Chung, Johnson, & Schill, 2006). No relevant empirical discussion on downside risk has been presented. The aim of the Estrada study (Estrada, 2007) is to determine the efficacy of downside risk in practice. Estrada's research motivation stems from the inefficacy of β which similar to

the motivation for this study. The shortcomings of β that are noted by Estrada are also similar to those noted by this study; for instance, the fact that higher upswings during good market conditions can result in higher values of β. While the β measure would imply that such investments are riskier, an investor would obviously disagree. As such, an investor may be concerned primarily with noting how volatile a security is when market conditions are *poor*, paving the way for the analysis of downside risk measures.

Estrada uses two separate measures of downside risk: the semi-variance or semi-deviation (discussed in section III) and the downside β. The downside β is defined as the expected percentage change in the returns on a security, given a 1% *decrease* of the market portfolio. The semi-variance and downside β are equivalent risk measures to variance and β but are computed only when the market is down. As such, it is a better method of evaluating how investors think about stock volatility. Aside from the risk measure choices, the Estrada study is also important owing to the broad nature of its data sample, incorporating data from developed and emerging nations and testing on them separately and collectively.

V.5.1 The Model

Remember that the aim of the Estrada study is to determine whether downside risk measures are useful in calculating stock returns. The author uses variations of the CAPM as the models to be tested for statistical significance. A general expected return-risk model is presented in equation 5.10 as follows:

$$RRE_i = R_f + MRP \cdot SR_i \tag{310}$$

where RRE_i is the expected return on equity for a company i, R_f is the risk-free rate, MRP is the market risk premium, and SR_i is the specific risk for i. In the traditional CAPM, the specific risk measure is the β of the stock. For the purposes of the Estrada study, the specific risk measure is replaced by the relative semi-deviation and the downside β (Estrada, 2006). This gives the models tested by the Estrada study, presented in equations 5.11 and 5.12 below,

$$RRE_i = R_f + MRP \cdot \left(\frac{\Sigma_{Bi}}{\Sigma_{BM}}\right) \tag{5.11}$$

$$RRE_i = R_f + MRP \cdot \beta_{Bi}^{D} \tag{5.12}$$

where Σ_{Bi} and Σ_{BM} represent the semi-deviation with respect to the benchmark return B for company i and the market respectively, and β_{Bi}^{D} represents the downside β with respect to the benchmark return B for company i. As such, Estrada's empirical model is as follows:

$$MR_i = \lambda_0 + \sum_{j=1}^{n} \lambda_i \cdot RV_{ji} + u_i \tag{5.13}$$

where MR is the mean return for company/country i, and RV_{ji} represents the risk measures utilized by the multiple regression equation and n is the number of risk measures included in the model.

V.5.2 Data and Methodology

To test the hypothesis that downside risk measures better predict returns than the β, the author uses data collected from the Morgan Stanley Capital Index (MSCI) database for 23 developing markets (DMs) and 27 emerging markets (EMs) for the time

period up to December 2001. For each of the markets, mean returns and four key risk measures are calculated: two from the mean-variance approach, i.e., σ and β, and two for the hypothesized mean-semivariance approach, i.e., Σ and β^D. The method for computing the semi-variance was shown in section III.3.1. The formula used to compute the downside β risk measure is as follows:

$$\beta_i^D = \frac{E\{\min[(R_i - \mu_i), 0] \cdot \min[(R_{Mkt} - \mu_{Mkt}), 0]\}}{E\{\min[(R_{Mkt} - \mu_{Mkt}), 0]^2\}} \tag{5.14}$$

The summary statistics for the data are provided in Table 11 of the Appendix. Note that the Estrada study uses a simple mean of monthly returns method to calculate target returns. This method is not as useful as using a geomean, as the true return for the holding period is computed by the geomean and not the simple average. This thesis uses a geomean method to compute the target returns.

Estrada tests his hypothesis using OLS regression. The results from the regression analysis are discussed in the next section. However, some of the regression tests demonstrate heteroskedasticity using the White test. For the regressions that are subject to heteroskedasticity, the heteroskedasticity-adjusted t-scores are reported. The following section discusses the results of the Estrada study and their implications for this thesis.

V.5.3 Results and Conclusion

Table 12 presents the results from the regression analysis conducted on the entire dataset. It is clear from the results in the table that even when adjusted for heteroskedasticity, the downside β measure is the best measure in explaining returns. Using the downside β measure provides the highest R^2-value and the most statistically significant outcome. While the semi-deviation outperforms the standard deviation and the β, the downside β outperforms even the standard deviation. The same results are obtained, even if a multiple variable regression analysis is conducted, and when the data is split into DM and EM and the regression process is repeated. For these results observe tables 13, 14, and 15.

The implication of the results on this thesis is that the downside beta will be used as the downside risk measure over the semi-variation. The reason is that the Estrada downside β outperforms the semi-variance measure, which was previously hypothesized as the measure for downside risk. The problem with the Estrada study is that it tests only downside risk measures in conjunction with the β. But Estrada does not take into account other risk variables, which may leave the study vulnerable to omitted variable bias. This thesis takes other risk measures into account such as higher order moments and leverage. The Estrada study reports only R^2-values. It should instead report adjusted-R^2 values as this accounts for overfitting by including additional variables. This study corrects for this in the results reporting and uses the geomean returns as explained in the previous section.

V.6 Laxmi C. Bhandari

So far the only variable for which empirical literature has not been reviewed is leverage. The Bhandari study (Bhandari, 1988) aims to gauge the explanatory power of leverage on returns, controlling for β and firm size. Bhandari uses the debt-equity ratio DER as the measure for leverage. Since the DER of a firm serves as a natural proxy for the risk associated with that firm's common equity, Bhandari proposes that DER may be used as an additional variable to explain common stock returns. This study also hypothesizes that leverage is important in explaining stock returns. As such, the results of the Bhandari study have important implications on this study as well.

V.6.1 The Model

Since investors are typically risk-averse, a positive correlation is expected between DER and expected returns from common stock. The following is Bhandari's empirical model:

$$\widetilde{r_{it}} = \widetilde{\gamma_{0t}} + \widetilde{\gamma_{1t}} \cdot LTEQ_i + \widetilde{\gamma_{2t}} \cdot BETA_i + \widetilde{\gamma_{3t}} \cdot DER_i + \widetilde{e_{it}} \tag{5.15}$$

where $i = 1, \dots, N$ represents the common stock i, $LTEQ_i$ is the natural logarithm of the market equity (in millions of Dec 1925 USD) of company i, $BETA_i$ is the calculated beta value for the firm i, and DER_i is firm i's debt-to-equity ratio. The following section describes the process used to estimate this equation.

V.6.2 Data and Methodology

The data used to conduct empirical analysis are taken from the CRSP database. In particular, data on shares outstanding, price per share, and nominal returns from the CRSP monthly stock returns files are collected. An equally weighted portfolio of all stocks on the NYSE is used as the market portfolio. The book values of total assets and common equity are collected from the COMPUSTAT database. Additionally, data on the Consumer Price Index (CPI) is collected from the Ibbotson-Sinquefield database so as to adjust the returns for inflation. Data is collected from 1946 (when COMPUSTAT began) onwards. The DER is calculated using the following formula:

$$DER = \frac{\text{book value of total assets - book value of common equity}}{\text{market value of common equity}}$$

Fama-Macbeth regression methodology, with some improvements that are beyond the scope of this study, is utilized to estimate the regression equation 5.15. The method utilized to compute β values is similar to Fama-Macbeth. Bhandari uses two-year subperiods to estimate β values. A similar size ranking followed by β ranking followed by a DER ranking procedure is used to arrive at 27 groups/portfolios that are used in testing by assuming that the individual stocks within each portfolio are equally weighted. The results from the regression analysis are presented in the next section.

V.6.3 Results and Conclusion

The results from Bhandari regression analysis are available from Table 16 in the Appendix. Note that in all cases, whether the whole list of firms is used or only manufacturing firms are used, the DER is the most important variable in determining returns, even in the presence of size and β. As such there is empirical evidence that suggests that the DER may be important in determining returns. We utilize the DER as our measure for leverage in this study.

Note that the Bhandari study is also open to omitted variable bias as it fails to take in to account several of the risk variables hypothesized by this paper. In this study, we take a multiple regression approach to understanding the effectiveness of alternative risk measures, which may result in varied results from the Bhandari study.

V.7 Concluding Remarks

In this section we summarize some of the key findings from the literature review. Recall that the aim of the literature review was to develop an empirical testing methodology that is optimal given our time and data restrictions. The following binds together a variety of elements that are key to developing this empirical testing methodology. Firstly, all the empirical work discussed in the prior sections has focused on specific alternate risk measures. As such, each of the empirical studies is open to omitted variable bias. This study is different in that it aims to analyze several alternate risk measures and determine how they interact with one another.

The Fama-French study backed this paper's theoretical claims that β and size were important in determining returns. In addition, it provides us with an additional important risk variable: the price-to-book ratio. We add this variable to our empirical discussion owing to the variable's strong empirical significance in the Fama-French study. In addition, it was determined from the Chung, Johnson and Schill study that higher order moments were also effective in determining returns. However, due to the lack of a theoretical basis, we restrict our scope to the third and fourth degree moments: skewness and kurtosis. The Bhandari study also provides support for using the debt-equity ratio as a measure of leverage.

While these studies were reviewed owing to the variables they tested, the Fama-Macbeth and Black, Scholes and Jensen studies were reviewed owing to the methodologies of testing that they presented. Both studies were conducted to test the efficacy of the CAPM. The Fama-Macbeth analysis utilizes a cross sectional regression methodology, a simplified version of which is used by this study. We do not construct any portfolios, as the data is insufficient to allow for both portfolio construction and to observe concrete relationships. A simplified version of the portfolio methodology presented by Black, Scholes and Jensen is utilized by this study to test for the explanatory power of alternative risk measures. The next chapter presents the empirical analysis conducted by this study.

VI. Empirical Analysis

So far in this study the theoretical basis for analyzing alternative risk measures has been highlighted followed by a discussion of prior empirical research conducted on alternative risk measures. In this chapter, the actual empirical testing of the effectiveness of alternative risk measures is presented and discussed. Chapter VI begins with a brief recap of the hypotheses being tested by the paper in section VI.1. Section VI.2 is a discourse on the data and methodology utilized by this study. This section also contains a discussion on the assumptions and limitations of the empirical models. Section VI.3 presents the results obtained after constructing and simulating the empirical model. This is followed by a discussion of these results and its implications for future empirical research in section VI.4.

VI.I Hypothesis Recap

Before the discussion of the empirical model, it is important to first briefly recap the hypothesis of this study, so as to be clear about what the empirical model should be testing. In other words, knowledge of the hypotheses is important in constructing a *valid* empirical methodology. Recall that the aim of this paper is to determine whether alternative risk measures add to the explanatory power of β in explaining actual stock returns. This analysis may then shed light on the way investors think about risk. If the CAPM and its assumptions hold, then the only thing that should matter to an investor is

the β of the security or portfolio. As such, the only risk that an investor is compensated for is its market risk as measured by β. However, this result may not hold if returns are not normally distributed, suggesting that the mean and variance do not adequately describe all the information required to accurately determine the distribution. Since actual market data suggests that returns may not necessarily be distributed normally (Mandelbrot, 2004), it is imperative to then consider alternative risk measures that might better explain returns in addition to the security's β.

This paper hypothesizes that along with volatility, an investor may also take into account the chance of losing money or the downside risk, the probability of Black Swan events, the leverage of the firm, and the size effect. The following is the conceptual model for this study:

$$\text{Return} = \text{f [Volatility, Size, Leverage, Downside, Skewness, Tail Risk]} \qquad (32)$$

The conceptual model is operationalized (in order) as follows:

$$Return = f\left[\, \beta^{+}, ME^{-}, \frac{D^{+}}{E}, Downside\ \beta^{+}, Skewness^{+}, Kurtosis^{+} \,\right] \qquad (33)$$

where ME is the market value of the firm's equity and $\frac{D}{E}$ is the firm's debt-to-equity ratio. Based on this model, data is to be collected and a methodology is to be developed that reliably tests these hypotheses. The data used and the methodology of the empirical analysis is discussed in the next section.

VI.2 Data and Methodology

The previous section establishes the hypotheses that are asserted by this paper. The purpose of this section is to develop an empirical framework that is a valid test of these hypotheses. This section presents the empirical framework and begins with a discussion of the data that is collected and used in the empirical analysis in VI.2.1. Section VI.2.2 details the methodology for the hypothesis testing using multivariate regression analysis. To provide more statistical weight to these results, the hypothesis testing is conducted using another methodology called the *Portfolio Construction Method* that is presented in section VI.2.3. Section VI.2.4 highlights some assumptions and limitations of this approach before the results are discussed in VI.3.

VI.2.1 Data Collection and Risk Measure Calculation

To conduct the empirical analysis, the second step following the establishing of the operational model is to determine the methodology for collecting the data required for the regression and portfolio analysis. Owing to the time restrictions placed on completing this study and due to the lack of access to the CRSP Database, stock data were only collected from the New York Stock Exchange (NYSE).

A listing of companies on the NYSE from 2015 was obtained ("Companies listed on the New York Stock Exchange (A)," 2015) and placed into a Microsoft Excel Workbook. Several duplicate listings were found and removed to arrive at a final listing of 2704 NYSE companies. A random sample of 300 companies was selected from the

available 2704. This was done by assigning a random number between 0 and 1 to each of the 2704 companies by using Excel's RAND() command and then sorting the list based on these randomly assigned numbers. This created a randomly arranged listing of the NYSE companies, and the first 300 companies that matched the data criteria were selected allowing for a bias-free selection. If a company did not have data for the entire time period, the company was skipped and the next company in the random assortment was selected.

For a company to be selected, it had to have data on historical monthly closing stock prices from December 2002 to January 2015. The data are available from the Yahoo Finance database. The financial database YCharts is used to collect the price-to-book ratio, debt-equity ratio, and market capitalization from December 2013. The debt-equity ratio is used as measure for leverage, the market capitalization as a measure for size, and the book-to-market ratio is used owing to its significance in the Fama-French (Fama & French, 1992) study. If all the requisite data from Yahoo Finance are available, then the company is selected for analysis. This process is repeated until price data for 300 companies are obtained. While the price data could be obtained for all 300 companies, the YCharts data was not available for all companies in the 300-company shortlist. For the market capitalization, 246 observations were obtained, for the price-to-book ratio, 239 observations, and 227 debt-to-equity ratio observations.

Along with the price data for the companies, adjusted closing price data for a market index was also collected so as to calculate the moment-based risk measures. There were several possible choices for the market index, including the S&P 500, the

Russell 2000, and the Russell 3000. However, the Wilshire 5000 Total Market Index (W5000) was chosen as the representative market index. This is keeping in line with Roll's critique and is aimed at capturing as broad a measure of the market as possible. The W5000 is a market cap-weighted index that consists of over 6,700 publicly traded companies. A company is selected to be part of the W5000 if it meets the following three criteria: (1) the company has its headquarters in the US, (2) the stock is actively traded on a US stock exchange, and (3) the stock's pricing information is widely available to the public ("Wilshire 5000 Total Market Index (TMWX) Definition," n.d.). Given the broad nature of the W5000, it is the best index that may be used to track the overall movement of the US stock market. Note that this is similar to prior empirical literature, which uses all the available companies from the NYSE, NASDAQ, and AMEX. Since data for all these securities is unavailable, the W5000 is the best alternative to using the entire data set.

It is important to note that the closing prices that are collected for the 300 companies are not just the closing price but the closing price of the month adjusted for any distributions or corporate actions that may have occurred at any time prior to the next day's open ("Adjusted Closing Price Definition," n.d.). Since this adjusted price takes into account corporate actions such as splits, dividend payments, distributions, and rights offerings, this adjusted price provides a better estimate for the equity value and gives a more accurate returns value. After the historical adjusted stock prices ($P_{i,t}$) are collected, the historical returns from 2003 to 2014 are collected for any company i in time period t as follows:

$$R_{i,t} = \frac{P_{i,t}}{P_{i,t-1}} - 1 \tag{34}$$

Since price data is available for 146 months, 145 historical returns values are obtained for each of the 300 companies. The returns from 2003 to the end of 2013 are used to calculate the moment based risk measures such as the β, skewness, excess kurtosis, and downside β using the formulas that are discussed in section III.3. This provides one value for each of these risk measures and for each of the 300 available companies. Henceforth, we set N to be the total number of available companies (i.e. $N = 300$).

The remaining 12 returns values from 2014 are used to compute the geometric mean return for company i for the year 2014 as follows:

$$\bar{R}_i = \sqrt[12]{\prod_{m=1}^{12}(1 + R_{i,m})} - 1 \tag{35}$$

where m is used to represent the 12 months of 2014. This geomean return value is used as the dependent variable in both the regression and portfolio analysis methods described in the next sections.

VI.2.2 Regression Analysis Methodology

Regression analysis is used to determine the extent to which the variation in a dependent variable can be explained based on the variation in a group of independent variables. In terms of this study, the geomean return values from 2014 for the 300 available companies are used as data inputs for the dependent variables, which is the

aim of explanation. The various risk measures that are hypothesized are used as independent or *explanatory* variables. If all the hypothesized variables were important in explaining returns, the following linear, multivariate, regression equation would be used to predict returns:

$$\hat{R}_i = \hat{\alpha}_{0,i} + \hat{\alpha}_{1,i}\,\beta_i + \hat{\alpha}_{2,i}\,\gamma_i + \hat{\alpha}_{3,i}\,\kappa_i + \hat{\alpha}_{4,i}\,\bar{\beta}_i + \hat{\alpha}_{5,i}\,ME_i + \hat{\alpha}_{6,i}\,\left(\frac{D}{E}\right)_i + \hat{\alpha}_{7,i}\,\left(\frac{B}{V}\right)_i + \hat{\epsilon}_i \tag{36}$$

where for any company $i = 1 \dots 300$, β_i is the CAPM risk measure, γ_i is the skewness, κ_i is the excess kurtosis, $\bar{\beta}_i$ is the downside beta, ME_i is the market value, $\left(\frac{D}{E}\right)_i$ is the debt-equity ratio, and $\left(\frac{B}{V}\right)_i$ is the book-to-value ratio. $\hat{\epsilon}_i$ is the stochastic error term. $\hat{a}_{j,i}$ for $j = 0 \dots 7$, are the regression coefficients for a particular company i. The coefficients are averaged out to obtain the desired coefficients that express the relationship between returns and the risk measures. The coefficients form an important part of our analysis since they are used for setting up and conducting the required hypothesis testing. Based on equation 6.5, it is clear that the following hypotheses are to be tested:

1. $H_0: \hat{\alpha}_1 \leq 0, H_A: \hat{\alpha}_1 > 0$
2. $H_0: \hat{\alpha}_2 \leq 0, H_A: \hat{\alpha}_2 > 0$
3. $H_0: \hat{\alpha}_3 \leq 0, H_A: \hat{\alpha}_3 > 0$
4. $H_0: \hat{\alpha}_4 \leq 0, H_A: \hat{\alpha}_4 > 0$
5. $H_0: \hat{\alpha}_5 \geq 0, H_A: \hat{\alpha}_5 < 0$
6. $H_0: \hat{\alpha}_6 \leq 0, H_A: \hat{\alpha}_6 < 0$
7. $H_0: \hat{\alpha}_7 = 0, H_A: \hat{\alpha}_7 \neq 0$

The coefficients described above are estimated using an OLS regression using the statistics software STATA. The results obtained from the OLS regression are discussed in section VI.3.

After having estimated the regression equation 6.5, the hypotheses mentioned above must be tested to check for statistical evidence. We assume that the null hypotheses may be rejected in favor of the alternatives using an $\alpha = 0.05$ level of significance (i.e. with a 95% confidence level). Also, depending on the amount of variation in returns that can be explained by the independent variables, the independent variables can be ranked in terms of their statistical significance in explaining returns. This is done by computing the *t-scores* for each of the independent variables from the regression at the 95% confidence level. To calculate the t-scores, the *degrees of freedom (dof)* must first be calculated. The dof is just the total number of coefficients that must be estimated subtracted from the total available data points,

$$dof = N - no.of\ coefficients \tag{37}$$

$$dof = 300 - 8 = 292$$

After calculating the degrees of freedom, the t-score for each coefficient may be calculated as follows (Studenmund, 2014):

$$t_k = \frac{\hat{\alpha}_k - \alpha_{H_0}}{SE(\hat{\alpha}_k)} \tag{38}$$

where $SE(\hat{\alpha}_k)$ is the estimated standard error of the estimated coefficient $\hat{\alpha}_k$, and α_{H_0} is the border value that may be found from the null hypothesis. In our case the border value is always zero. Therefore, the t-score for any of the coefficients may simply be calculated as:

$$t_k = \frac{\hat{\alpha}_k}{SE(\hat{\alpha}_k)} \tag{39}$$

To determine whether a particular variable is statistically significant in explaining returns or not, the t-score computed for the coefficient of that variable is compared to a *critical value* t_c that depends on the degrees of freedom that was computed previously, the confidence level specified (95% in this case) and whether the test is one-sided or two-sided (except for the last hypothesis which is two-sided, all tests are one-sided). For this case, with one-sided tests and over 200 degrees of freedom, the critical t-value t_c is ≈ 1.645. To reject the null hypothesis, the coefficient must have the same sign as hypothesized, and:

$$|t_k| > t_c$$

$$\therefore |t_k| > 1.645 \tag{40}$$

This method will be utilized to determine the statistical significance of the independent variables. Based on the outcome of the hypothesis testing, the original aim of this paper may be achieved: it will be determined if there is enough statistical evidence to show that alternative risk measures do add explanatory power to the β in explaining returns and that the CAPM is incorrect in utilizing simply the β as a measure of risk.

VI.2.3 Portfolio Analysis Methodology

The regression analysis method is a conventional and effective (Fama & MacBeth, 1973) method of determining the risk-return relationship. However, as was discussed in section III.2, when analyzing securities, the diversifiable risk is not rewarded with a return in value. This is because an investor can simply put securities together in a

portfolio and diversify away this risk. Empirical studies have suggested that 20 securities or more are needed to adequately diversify away risk from investing in stocks. However, this number varies based on the risk measure in question. When analyzing skewness, the number rises to 30 stocks (Chong & Phillips, 2013). The portfolio analysis method analyzes the effect of alternative risk measures on portfolio returns but has the advantage of being able to analyze the effect of the risk measures after diversification. The disadvantage of this method, however, is that it cannot analyze multiple risk variables at the same time, as is possible with the multivariate regression model. The portfolio method will also provide an alternative method to validate the results, adding weight to the results if the two methods concur, and casting doubt on the regression results if the two methods differ.

In the portfolio method, the 300 NYSE companies are first ranked based on the risk measure being analyzed. The sorted list is then split into 10 portfolios, each containing 30 companies (provided data are available). The reason for taking a sample of 300 companies is that it allows for diversification of each individually constructed portfolio regardless of the risk measure since each portfolio contains 30 securities. The risk measures are calculated in an identical manner to the way they were calculated for the regression method. To calculate the geomeans for each portfolio, the returns for each portfolio for each month in 2014 are first calculated by taking the equally weighted average of the returns of the portfolio's constituent companies. The geomean for these 12 values is then calculated.

As such, there are 10 portfolios, ranked on the basis of a particular risk measure, that each has its own returns. If the risk measure did not play a role, then returns on the different portfolios should not be statistically different from one another. However, if the returns are different from one another, then there is statistical evidence for that particular risk measure in determining stock returns. For instance, to analyze the effect of excess kurtosis κ in determining stock returns, the 300 companies are ranked based on κ, then split into 10 portfolios, each containing 30 companies. The 10 portfolios are now ranked from low to high κ. Now the monthly returns for these portfolios are calculated for the year 2014. Using these 12 values, it is possible to calculate a geomean return for that specific portfolio. The geomeans are tested using the methodology described in the next paragraph for statistical difference, and depending on the results, it may be judged whether κ is important in explaining returns. Note that the risk measure and portfolio return for each portfolio is the simple mean of the values for each security. This assumes that each portfolio is equally weighted, an assumption that is used for the sake of simplicity and owing to prior empirical studies such as Fama-French.

To test whether portfolios that have been ranked based on a particular risk measure exhibit varied returns, the Spearman's rank correlation test is utilized. To conduct the test, 10 portfolios are constructed as mentioned above. Following this, the returns values and risk measure values are averaged out and ranked in ascending order. Since the portfolios are sorted based on one of the risk measures, the risk measure averages are ranked serially from 1 to 10. The correlation between the two rank

columns is then found to determine the strength of the relationship between the returns and the corresponding risk measure.

VI.3 Results and Analysis

In this section, the main results of empirical testing are presented along with an analysis of the results. In section VI.3.1, the data collected using the methodology described above, is analyzed for errors and cleaned so as to prevent biased results. In sections VI.3.2 and VI.3.3, the results from the regression analysis and portfolio analysis methods are presented respectively. As will be shown in these sections, there is a lack of statistical significance in the results obtained using both these methods. A hypothesis for the unexpected results is a possible structural break caused by the financial crisis that occurs during the time frame used to collect and compute this study's data. In section VI.3.4, The same empirical tests are carried out, using only data that is unadulterated by the financial crisis to determine if the hypotheses would hold in an alternate, less volatile, timeframe.

VI.3.1 Data Cleaning

Before conducting empirical testing using regression analysis or portfolio analysis, it is important to first analyze the data for any discrepancies or outliers. The presence of extreme values may sway the results of the regression analysis or may cause improper rankings in the portfolio method and as such, should be eliminated from analysis. This chapter will analyze the data collected using the methodology suggested

in section VI.2.1. The following is a summary of the collected data, obtained using the *summarize* command in STATA.

```
. summarize
```

Variable	Obs	Mean	Std. Dev.	Min	Max
Company	300	150.5	86.74676	1	300
Ticker	0				
Return	300	.0017563	.0255848	-.1365	.0768
Beta	300	1.146712	.6897642	-.0454	5.6802
Downside	300	1.162155	.7333745	-.4131	3.7654
Skew	300	.0986283	1.114455	-5.0355	11.5223
Kurtosis	300	4.172965	9.047754	-.4457	132.8417
MEinbillions	246	22.88471	70.34123	.0483	844.74
PricetoBook	239	19.98778	250.1079	.2717	3867.65
DE	227	.9543683	1.92077	-7.486	20.5

Table 6.1: Summary statistics for un-cleaned data set

The 2014 returns values for the 300 companies labeled ‘Return’ in Table 6.1 seems to contain plausible return values. The values range from -13.65% to 7.68%. The values for the 300 βs (‘Beta’) range from -0.0454 to 5.6802. The following plot shows the distribution of the risk measure values.

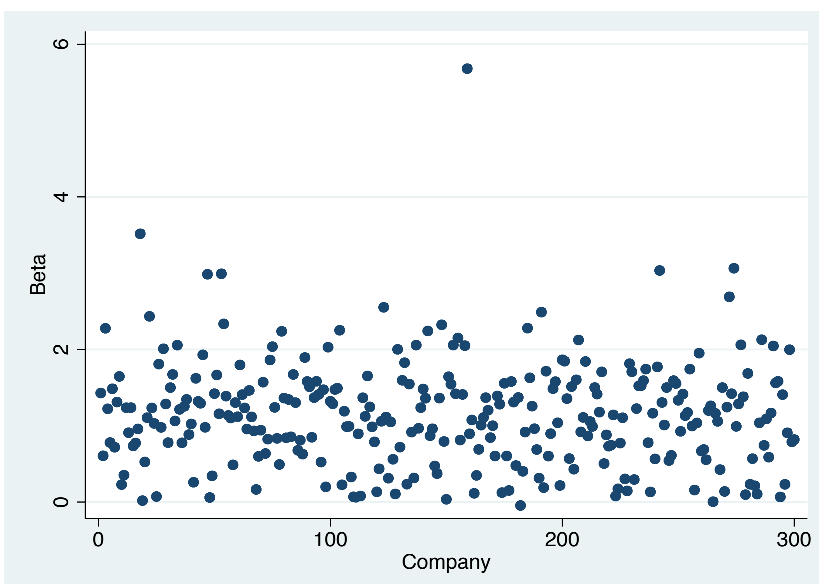

Figure 6.1: Plot of β by company

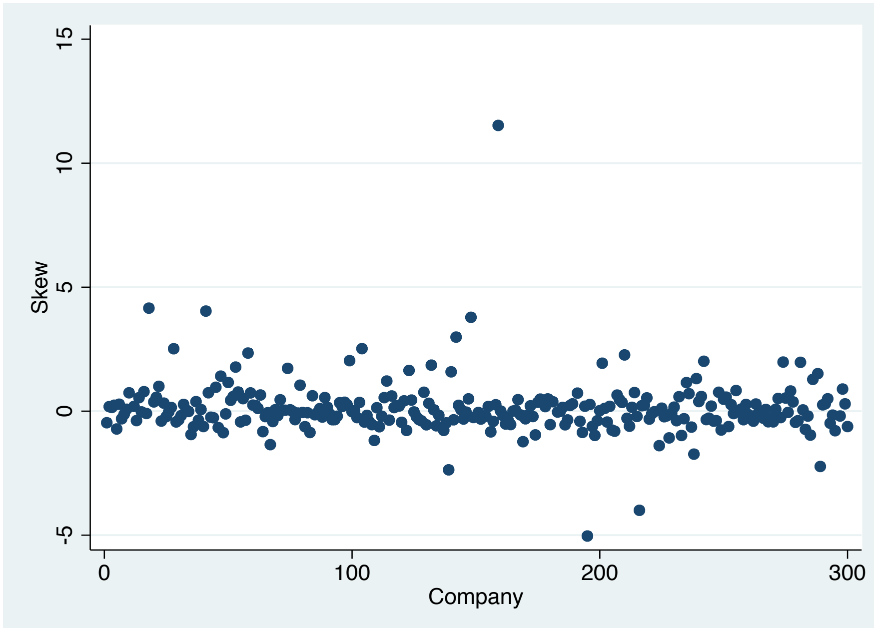

Figure 6.2: Plot of Skewness by company

It is clear from Figure 6.1 that there is only one β value that is negative. This is from the company Nuveen Municipal Income Fund, Inc. While there are some relatively low beta values, NMI is the only company that has a negative, albeit small value. Due to the unlikelihood of companies possessing a negative β value, this data entry is a little unrealistic. As such, it is omitted when hypothesis testing using regression as well as removed for conducting portfolio analysis. Negative values are similarly handled with respect to the downside β variable ('Downside'). Also note the presence of an outlier in the Beta plot. One company has a β value close to 6, much higher than the other values in this range. This company itself is an outlier and is dropped from all analyses due to its extremely high risk-measure values, not only of β, but also of skewness and kurtosis as is evident from the following plots. Note that it is the same company that is the clear

outlier in Figures 6.1, 6.2, and 6.3. This company is Telephone and Data Systems, Inc. (TDS), and the reason for the extreme values is that the company underwent a stock split in May 2005.

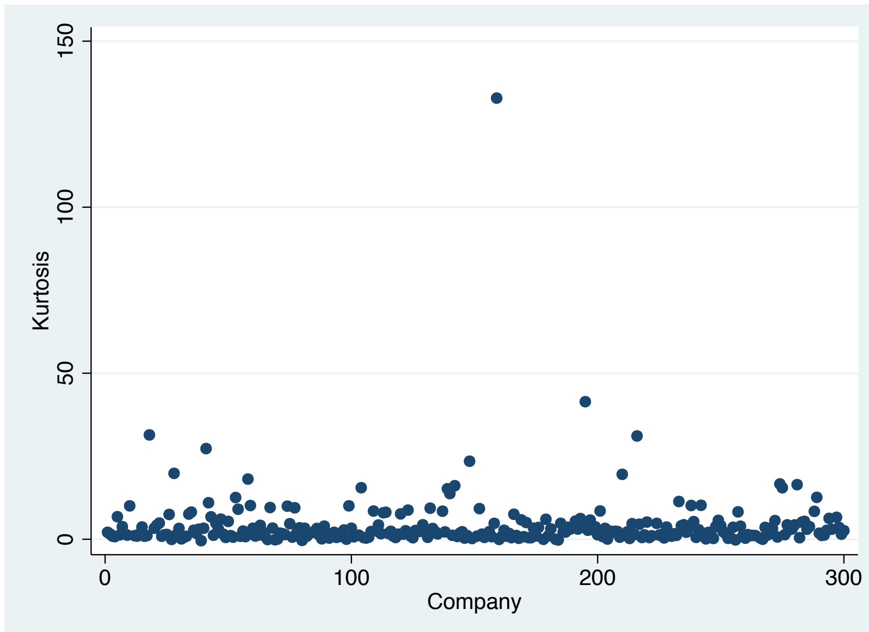

Figure 6.3: Plot of Excess Kurtosis by company

Additionally, some other data points contained unusable data for a variety of reasons including extreme returns variation following stock splits, error in collection, or some mechanical oversight, resulting in outlying values of beta, skew, or kurtosis. These data points were found and removed. The companies removed were: GPM, KEM, and ITT. Note that outlier data was not removed from $ME, \frac{D}{E}$, and price-to-book values since these values were simply taken from the YCharts database and represents no flaw in methodology. There were some errors in collection, which were amended. The updated summary statistics, with all of the aforementioned removals is shown in Table 6.2 below.

```
. summarize
```

Variable	Obs	Mean	Std. Dev.	Min	Max
Company	285	149.5649	86.55048	1	300
Ticker	0				
Return	285	.0017726	.0259657	-.1365	.0768
Beta	285	1.15982	.6097523	.059	3.0638
Downside	285	1.212253	.7026239	.043	3.7654
Skew	285	.0902744	.790528	-2.3671	4.036
Kurtosis	285	3.480967	4.133865	-.4457	27.2816
MEinbillions	238	17.10887	36.53014	.0483	282.01
PricetoBook	231	20.6092	254.3975	.2717	3867.65
DE	219	.9543333	1.944897	-7.486	20.5

Table 6.2: Summary statistics for cleaned data set

The dataset so obtained is now ready for analysis. It is, however, important to note that due to the removal of 15 data points, the sample size is not only smaller, but also less random in terms of the selection. This may cause some slight dilution to the validity of the results. However, it is likely that the negative effects caused by including them in the sample far outweigh the positives of keeping these outlying data points.

VI.3.2 Regression Analysis Results

Regression analysis was conducted on the dataset described in the previous section to determine the validity of the hypothesized variables as alternative risk measures. The regression analysis began with a simple regression of returns on β, and continued to add variables depending on their significance in explaining the returns. The following tables show the results (coefficients and t-scores) of the regression analysis:

		COEFFICIENT VALUES						
Experiment	R-Squared	Beta	Downside	Skewness	Kurtosis	ME	D/E Ratio	Price-to-Book
1	0.0218	-0.006765	-	-	-	-	-	-
2	0.0311	-0.016347	0.009297	-	-	-	-	-
3	0.0284	-0.015175	0.008745	-0.001062	-	-	-	-
4	0.0277	-0.016369	0.009244	-	0.000058	-	-	-
5	0.0080	-0.014156	0.008526	-	-	0.000003	-	-
6	0.0189	-0.017264	0.010265	-	-	-	-0.000206	-
7	0.0122	-0.013103	0.006967	-	-	-	-	-0.000009
8	0.0055	-0.005796	-	-	-	-0.000002	-	-0.000010
9	0.0049	-	-0.003391	-	-	-	-	-
10	0.0100	-	-0.002448	-0.004268	0.000404	-	-	-
11	0.0103	-	-	-0.003862	-	-	-	-
12	0.0095	-	-	-0.004835	0.000371	-	-	-
13	0.0044	-	-	-0.004097	-	0.000001	-	-
14	-0.0032	-	-	-0.002545	-	-	-0.000576	-
15	0.0178	-	-	-0.005323	-	-	-	-0.000008
16	-0.0033	-	-	-	-0.000092	-	-	-
17	-0.0039	-	-	-	-	0.000013	-	-
18	-0.0031	-	-	-	-	-	-0.000550	-
19	0.0013	-	-	-	-	-	-	-0.000008
20	0.0080	-0.014416	0.006989	-0.001084	0.000455	-0.000016	-0.000900	0.000561

Table 6.3: Regression analysis results (adjusted-R^2 and variable coefficients)

Table 6.3 displays the coefficient values for the estimated regression equations. These coefficients may be easily interpreted. For instance, from experiment 1, the coefficient value of β is -0.006765, which implies that on average, when the β for a stock increases by one unit holding all other variables constant, the returns from stocks decreases by 0.6765%. As another example, consider the coefficient of ME from experiment 5. The coefficient value is 0.00003, which implies that on average when market cap for a stock went up by $1 billion holding all other variables constant, its returns increased by 0.003%. The t-scores presented in Table 6.4 are also easily interpreted. The value for the t-score should appear with its hypothesized sign and should exceed 1.645 in absolute value.

		T-SCORES						
Experiment	R-Squared	Beta	Downside	Skewness	Kurtosis	ME	D/E Ratio	Price-to-Book
1	0.0218	-2.71	-	-	-	-	-	-
2	0.0311	-2.94	1.93	-	-	-	-	-
3	0.0284	-2.50	1.76	-0.48	-	-	-	-
4	0.0277	-2.94	1.91	-	0.15	-	-	-
5	0.0080	-2.12	1.55	-	-	0.06	-	-
6	0.0189	-2.51	1.8	-	-	-	-0.22	-
7	0.0122	-1.93	1.24	-	-	-	-	-1.27
8	0.0055	-1.70	-	-	-	-0.03	-	-1.33
9	0.0049	-	-1.55	-	-	-	-	-
10	0.0100	-	-1.07	-1.85	0.94	-	-	-
11	0.0103	-	-	-1.99	-	-	-	-
12	0.0095	-	-	-2.16	0.87	-	-	-
13	0.0044	-	-	-1.72	-	0.01	-	-
14	-0.0032	-	-	-0.98	-	-	-0.61	-
15	0.0178	-	-	-2.20	-	-	-	-1.19
16	-0.0033	-	-	-	-0.25	-	-	-
17	-0.0039	-	-	-	-	0.27	-	-
18	-0.0031	-	-	-	-	-	-0.58	-
19	0.0013	-	-	-	-	-	-	-1.14
20	0.0080	-1.80	1.12	-0.31	0.75	-0.32	-0.79	1.31

Table 6.4: Regression analysis results (adjusted-R^2 and t-scores)

As is clear from the tables above, 20 experiments were performed to determine which variables were important in explaining returns. As is evident from the low adjusted R^2 values, none of the models that were estimated using OLS regression were successful in adequately explaining returns. Adjusted-R^2 values are used because this adjusts the increase in the degree of fit for overfitting by the inclusion of additional variables. The only variables that consistently maintained statistical significance were the β and the downside β of the securities, but appeared with a negative sign; this was an unexpected outcome. A negative β coefficient implies that the returns decreased on average as the β of the stocks increased, a clear breach of the intuition behind a direct risk-return relationship.

This result does not necessarily imply that Sharpe's theory is incorrect and that β works in the opposite way from the CAPM; it could be the case that it is this particular

dataset and methodology produced these seemingly strange outcomes. For instance, consider the results from experiment 8. Experiment 8 is the Fama and French Three Factor Model, discussed in V.1. Note that in the presence of ME and price-to-book value, β is still the most significant variable. None of the three variables have their expected signs, nor are they statistically significant. This result is surprising since the same variables were statistically significant in the Fama-French study, which followed a similar cross-sectional regression methodology.

It is interesting to note that despite the fact that the negative β relationship goes against common theoretical knowledge of return and risk, this result is supported by the empirical findings of Black, Scholes and Jensen (2006) that was discussed in the literature review. In that study too, it was noted that higher β values seemed to empirically correspond to lower returns and vice-versa. The fact that other empirical studies have also found a similar pattern suggests that it might be beneficial for future empirical work to investigate this relationship more closely.

Returning to the discussion on results, it is noted that in the presence of β, downside β appears as a statistically significant variable. However, while already using β, there is no theoretical basis for using downside as well, as the two measures are similar in terms of what they represent. In the absence of β, the downside β also becomes negative and suggests that the model is highly volatile to changes in variables and conditions. Skewness is another variable that appears to be statistically significant but still appears with its sign reversed as is evident from experiment 11. However, the t-

scores fluctuate when other variables are added, suggesting once again that the results are volatile and are not to be taken as being demonstrative of explanatory power.

Under these unintuitive circumstances, it is important to theorize what the reasons might be for the results presented in Table 6.4. It might be the case that the data collected is insufficient to smooth out the effect of outlying values and negate random effects. However, the existence of over 200 data points on most regression occasions is well beyond the traditional number of 25 samples. Of course, one negative aspect of the cross-sectional regression approach is the inability to separate diversifiable risk from the risk encapsulated by the hypothesized variables. As such, the next section may better elucidate the risk-return relationships that are theorized by this study. Yet another more plausible reason is that a structural break may have occurred during the time period of the data sample: namely the 2007-2009 financial crisis. During this crisis, high volatility in the stock market, coupled with low investor confidence may have resulted in large deviations in the values of returns and therefore, in the calculations of the risk measures. This will be further discussed in section VI.3.4.

To validate that the oddity in the regression results was not caused by a fault in methodology, consider Figure 6.4 below. Figure 6.4 plots returns against the β values, and as is evident from the plot, there seems to be no distinctly visible relationship between returns and β. In fact, it is reasonable that the line of best fit be negative, as even if a relationship were to be extrapolated from the data, it would match the data's slight, negatively sloped, tilt.

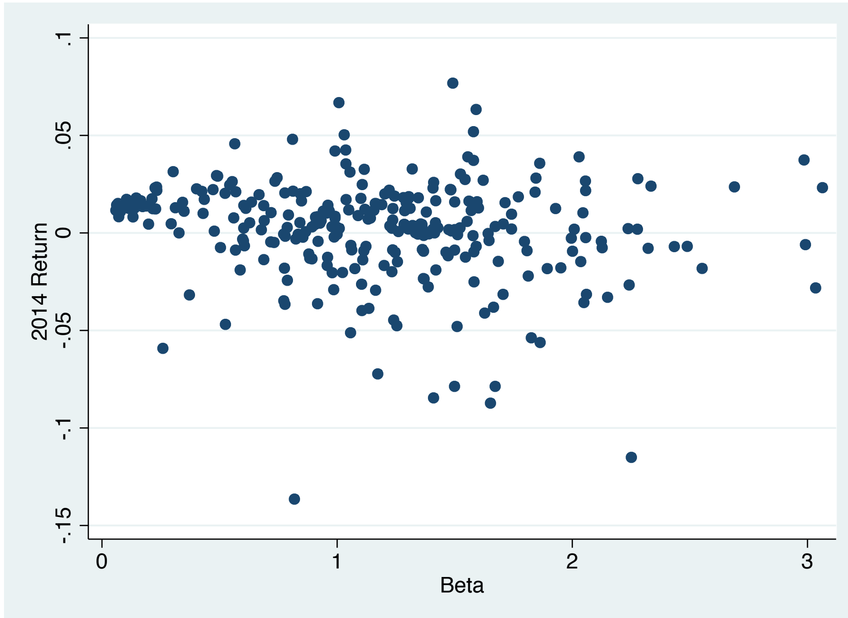

Figure 6.4: Plot of Geomean 2014 returns against β

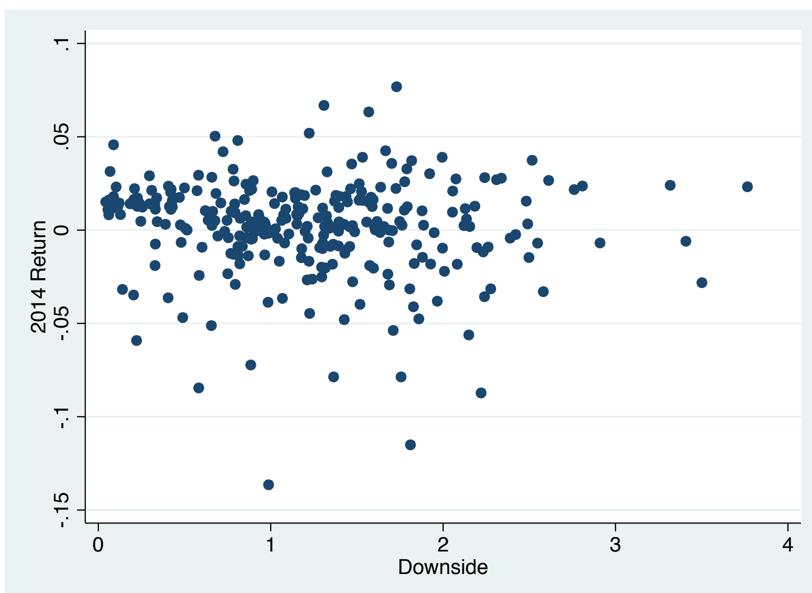

Figure 6.5: Plot of Geomean 2014 returns against downside β

Consider the plot of return values against downside β, as exhibited in Figure 6.5 below. Once again it is impossible to claim any kind of significant dependency between the two variables and suggests that the extremely low R^2-values and lack of statistical significance arise not from a flaw in methodology but from the nature of the data itself.

The skewness, which was the only variable to appear with both the hypothesized relation as well as slight statistical significance, also does not seem to show any concrete form of dependency, as shown in Figure 6.6.

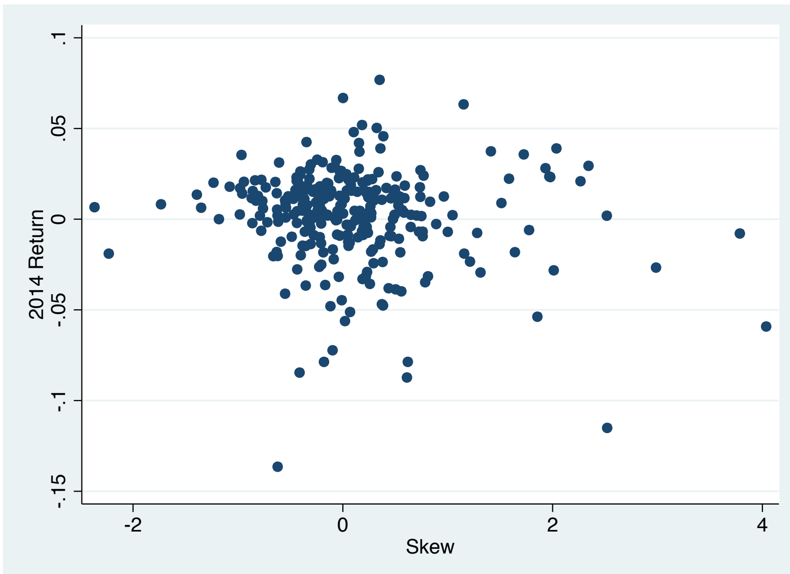

Figure 6.6: Plot of Geomean 2014 returns against Skewness

Given the failure of the regression methodology to provide any statistically significant explanatory risk measures for returns, we turn to the portfolio method to see if any such explanatory variables may be found. It may also be noted that no regression diagnostics tests such as tests for multicollinearity and heteroskedasticity (serial correlation is more common in time series) were conducted. The reason for this was the lack of any statistically significant model and the failure of the model to reject the null hypotheses. No new information is to be gained by testing for regression issues that will further decrease t-values.

VI.3.3 Portfolio Analysis Results

Portfolio analysis was also conducted on the dataset using the methodology described in section VI.2.3. As mentioned in the methodology, to test for statistical significance, Spearman's rank correlation test is used. The results presented in this section are simple correlation coefficients, and are therefore easily interpreted. The correlation coefficients for the rank correlation tests are provided in Table 6.5 below:

Risk Measure	Rank Correlation with Returns
Beta	-0.6606
Downside Beta	-0.6121
Skewness	-0.4182
Kurtosis	0.2485
Market Cap	0.6121
Debt/Equity Ratio	0.0182

Table 6.5: Portfolio rank correlations

It is clear from the table above that even after diversifying away risk, the risk-return relationship is still skewed away from theoretical expectations. The correlation with β is still negative, even after ranking them based on β values. This suggests that in the data sample, higher βs are associated with *lower* returns, a non-intuitive and unexpected result (but still conforming to the empirical work conducted by Black, Scholes and Jensen). The downside β rank correlation value, while maintaining a relatively high magnitude, also possesses the unexpected negative relation. The Market Cap variable also follows this trend of having a high magnitude but an opposite sign. It was hypothesized that smaller firms would generate higher returns (negative

relationship), but clearly on the basis of this analysis, the empirical data does not exhibit this theoretical prediction. The results for the skewness rank correlation tests mirror the results from the regression analysis. When ranked for increasing skewness, the portfolios do exhibit somewhat varied returns and in the direction hypothesized by the study. However, as mentioned earlier, it seems that the sample risk measures and return values arise from an extremely volatile economic time, and as such, the results for the portfolio tests also seem to be deviated from their regular expectations, as may hold in relatively normal economic climates.

In the following section, the data collected is analyzed in a similar fashion, but in a manner that isolates normal economic conditions. As such, it may provide a better understanding of why the results obtained from regression analysis and portfolio analysis is unexpected.

VI.3.4 Financial Crisis: Results and Analysis

The recently concluded financial crisis ('The Great Recession') had a widespread impact on the GDP, growth, employment, and overall economy of the US. As the housing bubble burst, the nation was plunged into the worst economic recession since the Great Depression. Owing to this, stock markets tumbled and investor confidence was shattered. As with any turbulent economic state, the stock market was highly volatile during this period. Consider the following plot of the monthly returns of the Wilshire 5000 Total Market Index from 2003 to 2015.

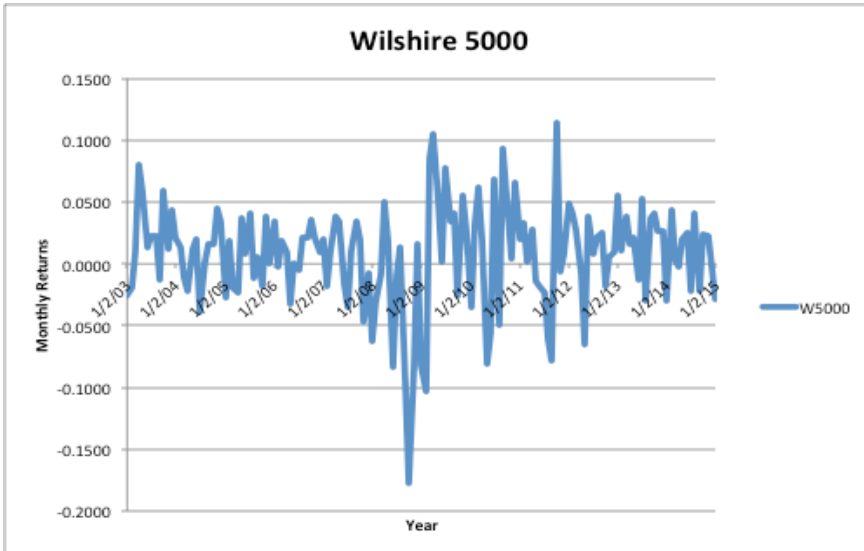

Figure 6.7: Returns on the W5000 – 2003 to 2014

It is clear from Figure 6.7 that the returns data was most volatile from the beginning of the financial crisis (approximately dated December 2007) to its end in mid-2009. As the market readjusted, returns values were volatile well into 2012. It is, perhaps, owing to this market turbulence that the values of the risk measures calculated in the preceding sections, are not able to predict returns values in 2014. It may be the case that the market was simply too turbulent, resulting in data which did not truly reflect companies' underlying risk values.

To test for a structural break, two relatively normal economic periods were constructed with less turbulent economic conditions. As is evident from Figure 6.7 above, the two most stable economic periods in the data time range were from January, 2003 to January, 2008 and from January, 2012 to January, 2015. These two periods were selected to test for the structural, representing pre financial crisis (PreFC) and post financial crisis (PostFC) data sets respectively. The target returns for the PreFC data set were set as the geomean of the 12 return values of the company in consideration from

the year 2007. For the PostFC dataset, these target returns were set as the geomean of the 12 return values from 2014. The following tables show the summary statistics for the two datasets:

```
. summarize
```

Variable	Obs	Mean	Std. Dev.	Min	Max
Company	290	150.0379	87.44692	1	300
Ticker	0				
Return	290	-.0037352	.0259636	-.0935	.0747
Beta	290	1.088563	.6750024	.0105	4.3183
Downside	290	1.024244	1.675581	-3.2362	8.5084
Skew	290	-.0133745	.8084728	-2.4233	3.8617
Kurtosis	290	1.322819	2.26431	-1.1144	19.76

Table 6.6: PreFC Summary Statistics

```
. summarize
```

Variable	Obs	Mean	Std. Dev.	Min	Max
Company	270	148.8889	87.51494	1	300
Ticker	0				
Return	270	.0011663	.0263424	-.1365	.0768
Beta	270	1.124501	.6674076	.0027	3.7919
Downside	270	1.360041	1.54926	-2.3883	5.7417
Skew	270	-.0638056	.6217717	-2.3599	2.1815
Kurtosis	270	.4684759	1.471571	-1.4607	8.2524

Table 6.7: PostFC Summary Statistics

The dataset shown above has been cleaned in a manner similar to section VI.3.1. It contains no outlying values and is ready for testing. Owing to time limitations, data for the $ME, \frac{D}{E}$, and price-to-book value variables have not been collected. These variables are not as important to this testing set as the prior full data range. This is because the aim of this empirical testing section is to identify whether a structural break caused by

the financial crisis is responsible for the lack of significant empirical results in section VI.3.2 and VI.3.3. To test for variable significance we follow the same procedure as described in VI.2. To check for variable significance in the PreFC model, consider the t-scores presented below:

		T-SCORES			
Experiment	R-Squared	Beta	Downside	Skewness	Kurtosis
1	0.0127	-2.17	-	-	-
2	0.0377	-3.24	-	2.91	-
3	0.0186	-2.38	-	-	-1.65
4	0.0460	-3.50	-	3.04	-1.87
5	0.0128	-	-2.18	-	-
6	0.0213	-	-2.35	1.87	-
7	0.0230	-	-2.29	1.87	-1.23
8	0.0059	-	-	1.64	-
9	0.0086	-	-	1.65	-1.34
10	0.0026	-	-	-	-1.33
11	0.0529	-3.17	-1.75	3.07	-1.74

Table 6.8: Regression results (t-scores) for PreFC data

It is evident from the data presented in Table 6.8 above that the split into pre and post financial crisis data sets still leads to unintuitive results. It is clear from experiments 1 to 7 that β and the downside β are still statistically significant in explaining returns. However, they still appear with the opposite (negative) sign as was found in the previous results. In fact the negative relationship between these variables and returns appears to be *stronger*, pre financial crisis. An important outcome is the statistical significance of the skewness variable. In the PreFC dataset, this variable is both statistically significant and appears with its expected (positive) relationship. In the presence of other variables, the statistical significance of skewness increases and the overall model seems to perform better. The excess kurtosis is still statistically

insignificant in all experiments. The following table presents the t-scores from the PostFC dataset analysis:

		T-SCORES			
Experiment	R-Squared	Beta	Downside	Skewness	Kurtosis
1	0.0447	-3.69	-	-	-
2	0.0411	-3.67	-	0.00	-
3	0.0414	-3.55	-	-	-0.30
4	0.0378	-3.54	-	-0.03	-0.30
5	0.0544	-	-4.06	-	-
6	0.0519	-	-4.09	-0.54	-
7	0.0486	-	-3.96	-0.55	-0.29
8	-0.0036	-	-	-0.17	-
9	-0.0036	-	-	-0.23	-1.00
10	-0.0001	-	-	-	-0.99
11	0.0626	-2.23	-2.84	-0.34	0.02

Table 6.9: Regression results (t-scores) for PostFC data

The regression results from the PostFC data set possess similarly poor R^2 values to the PreFC data set, but nonetheless show an improvement in fit from the R^2 values obtained through the analysis conducted on the entire dataset. Once again, both the β and the downside β are statistically significant in explaining returns but appear as a negative relation to returns. The negative relationship between β/downside β and returns is the strongest in the PostFC data set as compared to the entire dataset or even the PreFC data set. This may suggest that investors have become more risk-averse following the Great Recession, demanding less higher risk securities. An interesting outcome from the analysis conducted on the PostFC data set is that the statistically significant dependency of returns on skewness seems to disappear after the financial crisis, as is evident from experiments 2, 4, 6, 7, 8, and 9 in Table 6.9 above. As in the

other cases, excess kurtosis is still statistically insignificant and does not seem to explain returns.

Using the portfolio method does not change the nature of the results either. This is evident from the correlation values obtained in the following tables:

Risk Measure	Rank Correlation with Returns
Beta	-0.5030
Downside Beta	-0.5636
Skewness	0.6606
Kurtosis	-0.5394

Table 6.10: Portfolio Rank Correlations - PreFC

Risk Measure	Rank Correlation with Returns
Beta	-0.9879
Downside Beta	-0.8303
Skewness	-0.3697
Kurtosis	-0.2970

Table 6.11: Portfolio Rank Correlations – Post FC

As observed from the rank correlation values, the portfolio analysis seems to adhere to the results obtained from the regression analysis. As was the case with the regression analysis results, even after diversification the relationship between returns and β/downside β is observed to be negative. Skewness exhibits a strong positive relation to returns prior to the financial crisis and this relationship disappears after the crisis. Kurtosis is shown to be statistically insignificant in explaining returns. An interesting result from the portfolio analysis, however, is the extremely high negative rank correlation between the returns and β measures, post 2011. The portfolios created by ranking according to the β measures nearly matched a perfect negative correlation,

meaning that the lowest betas generated the highest returns and so on. This result is intriguing and may suggest that investors became significantly more risk averse after the financial crisis, and the demand for securities with higher betas significantly reduced.

Based on the results presented above, it can be concluded that the volatility of the stock market during the financial crisis is not the cause of the unexpected results obtained in sections VI.2.2 and VI.2.3. Clearly, the negative relationship between β and returns exists, prior to and even after the financial crisis, and indeed seems to be even stronger in these two time periods. As such, the cause of the unexpected results may actually have something to do with the methodology process itself, instead of the economic climate. In particular, the unconventional (but theoretically reasonable) choice of market index may be causing the deviation in results. The next section empirically tests whether the choice of market index is causing these unexpected results.

VI.3.5 Varying Market Index: Results and Analysis

In this section, the market index is changed from the Wilshire 5000 to a more conventional market measure which is the Standard and Poor's 500 (SP500). Similar to the previous section, regression analysis is conducted on the new data set using the SP500 as the market measure instead of the W5000. This causes changes in the values of the calculated risk measures, i.e. β, downside β, skewness and excess kurtosis. Similar to the previous section, the analysis is conducted for just the aforementioned calculated risk measures since this section does not test for the statistical significance of

the risk measures but tests instead for the reason why the primary methodology failed to produce the expected results. The following table shows the t-scores from regression experiments conducted on the full data set but using the SP500 as the market measure:

		T-SCORES			
Experiment	R-Squared	Beta	Downside	Skewness	Kurtosis
1	0.0179	-2.48	-	-	-
2	0.0180	-1.80	-	-1.01	-
3	0.0148	-2.50	-	-	0.33
4	0.0174	-1.77	-	-1.33	0.92
5	0.0054	-	-1.60	-	-
6	0.0105	-	-1.03	-1.56	-
7	0.0106	-	-1.08	-1.86	1.02
8	0.0102	-	-	-1.98	-
9	0.0100	-	-	-2.20	0.96
10	-0.0034	-	-	-	-0.16
11	0.0184	-1.80	1.13	-0.99	0.71

Table 6.12: Regression results (t-scores) for SP500 data

As is evident from Table 6.12 above, the results presented in Table 6.4 remain consistent, even if the market measure is changed. In fact, based on the t-scores of the included variables, it is also evident that the variables calculated using the W5000 index are better at explaining returns than those calculated using the SP500. However, the main outcome from this analysis is that the relationship between returns and β/downside β remains negative, regardless of the overall economic climate or the market index measure.

VI.4 Conclusion and Scope for Future Work

It is clear from the results presented in section VI.3, that either the assumptions made by the methodology of this paper are too lax to allow for empirically significant results, or that there really is no statistically significant relationship between the alternative risk measures proposed by this study and the returns generated by investing in a company's stock. Additionally, the results obtained in this study do not conform to the general literature consensus (excluding the Black, Scholes and Jensen study) regarding alternative risk measures. The empirical results obtained in this section, while unexpected, have remained consistent through testing with various time periods and market indices suggesting that if the methodology is appropriate then the conventional literature with regards to these risk measure is at the least open to some debate. It is important to note that the study's results with respect to β seem to match the results found by the Black, Scholes and Jensen study. Both empirical studies find a negative relationship between β and returns. If these results are empirically valid, then it suggests that the general consensus of β as a risk measure may be incorrect, as it seems to underestimate the risk-averse nature of investors, especially in a troubled financial climate as has prevailed in the US since 2008.

All of these questions may be answered by conducting further empirical analysis that lies beyond the scope of this study. Based on the results we have obtained, it is now possible to answer the question that this study aimed to examine. Empirical evidence suggests that alternative risk measures *do not* add any explanatory power to

the β of a stock, but the positive relationship between β and returns, as hypothesized by the CAPM, is still not an unquestionable financial truth, owing to a statistically determined negative relation. However, it is important to note some limitations with the empirical testing methodology.

Firstly, empirical literature such as Fama and French has suggested that it is better to assign β values for the regression analysis by portfolio construction. However, owing to the limited data collected (only 300 companies from NYSE), this method was not feasible. Further research may include a larger data set, particularly using the CRSP database, allowing for size-ranked β values to be assigned to individual securities. Additionally, more sophisticated testing methods than a simple rank correlation test could be used for the portfolio methodology. Note also that to test for the significance of the results obtained, other empirical tests may be conducted. For instance, risk measure values could be restricted to within two standard deviations to prevent any strong outliers from significantly affecting results. In the portfolio method, an additional experiment could be performed by optimally weighting the securities in each portfolio. In this study, for simplicity equal weights are assumed for each stock in a portfolio. Also portfolios should updated as time progresses as it is unrealistic to assume that stocks would remain in the same portfolio for the entire time period.

There are also some alternative risk measures that were not accounted for by this study such as semi-variance and value-at-risk. This leaves the regression model susceptible to some level of omitted variable bias. Future research may include such variables to analyze their impact on returns. There is also some level of survivorship bias

to which this empirical study is open, an issue that may be addressed by future studies. It may also make for a better empirical methodology to include stocks from other exchanges such as the AMEX and NASDAQ, or even international stock markets. In fact, it may be interesting to analyze these markets separately to see if the lack of significance of alternative risk measures is true only for the NYSE and whether changing the stock market under consideration changes the effect of alternative risk measures.

While it is true that this study has limitations with respect to its empirical methodology and that there is scope for future research, nevertheless this study contributes to research by conducting an analysis of several alternative risk measures together. It also tests its hypotheses using multiple methods and testing through a variety of time periods, suggesting that the results from this study are still useful with relation to the literature presently available on alternative risk measures.

VII. Conclusion

This thesis began with the aim of determining whether alternative risk measures such as higher-order moments, firm size, leverage, etc. added explanatory power to β when predicting stock returns. To this end, the study began with a theoretical foundation for why these alternative risk measures may be important in explaining returns. As part of the theoretical framework, we analyzed risk as viewed from a modern portfolio theory perspective. We discussed the traditionally held view that risk may be thought of as volatility and could be measured by β and that expected returns could be calculated using the Capital Asset Pricing Model (CAPM). The CAPM was then critiqued, showing that empirical literature does not conform to the assumption that stock returns are normally distributed. This formed a theoretical inspiration for pursuing alternative risk measures, hoping that these alternative risk measures may better explain returns.

The non-normality of returns is a clear factor that determines the effectiveness of alternative risk measures. As such, in Chapter IV we examined the nature of stock prices by assuming that returns were normally distributed. The outcome of returns being normally distributed was that returns could be mathematically modeled using a fair-game modeling system called a martingale process. Based on the properties associated with martingales, future researchers now have additional testable hypotheses to determine the normality of returns distributions.

In addition to the theoretical framework established in Chapters II, III, and IV, Chapter V reviewed prior empirical literature conducted on alternative risk measures. Several empirical methodologies were reviewed to identify the appropriate testing methodologies given the time limitations and data availability. Data collected from over 300 NYSE companies was used to conduct empirical analysis through two different methods: regression analysis and portfolio method. The methodology and results from the empirical analysis was presented in Chapter VI.

The results from the empirical analysis demonstrated a lack of statistical significance of alternative risk measures in predicting stock returns. In relation to central hypothesis of this paper, based on empirical analysis it was determined that alternative risk measures *did not* assist the conventional risk measure β in predicting stock returns. However, an interesting empirical outcome was regarding the β measure itself. The β (and on certain occasions the downside β), was the only empirically significant determinant of returns. Conventional literature with relation to the CAPM had theorized that there is a positive relationship between β and stock returns. However, the empirical results from this study mirror the empirical results from the Black, Scholes and Jensen (2006) study in that the empirical dependence of returns on β appeared to have a *negative* relationship.

While this outcome is unexpected from a conventional theoretical context, a reason for the negative relationship could arise from the fact that the financial crisis occurs during the time period of this study's data range. It is possible that during and after the financial crisis, investors may have become more risk-averse; choosing to avoid

stocks that possessed a higher β value, resulting in lower price increases and thereby lower returns for high-β stocks. However, upon further empirical investigation, this result seemed to hold prior to the financial crisis as well but strengthened following the financial crisis.

The results from this study can now be used to determine any suggested policies or changes that financial companies and investors may consider. There is no real empirical evidence to suggest that financial planners and investment companies should utilize alternative risk measures to calculate expected returns instead of conventional risk measures such as β or variance. However, the relation between β and returns is open to question. If the relationship is indeed negative instead of the positive relationship suggested by the CAPM, then there could be severe miscalculations in expected stock returns. While the results from this study are not conclusive, it paves the way for future analysis, which is critically important as several US firms and investors still use the β as the quintessential measure of risk. To conclude, it must be stated that questioning conventional literature is an important step to determine whether current business and investment practices are optimal. While this paper does not conclusively find any important alternative risk measures, the process of updating our investment knowledge and the search for a better risk measure should not be abandoned.

Appendix: Literature Review Tables

Fama and French

	All	Low-β	β-2	β-3	β-4	β-5	β-6	β-7	β-8	β-9	High-β
Panel A: Average Monthly Returns (in Percent)											
All	1.25	1.34	1.29	1.36	1.31	1.33	1.28	1.24	1.21	1.25	1.14
Small-ME	1.52	1.71	1.57	1.79	1.61	1.50	1.50	1.37	1.63	1.50	1.42
ME-2	1.29	1.25	1.42	1.36	1.39	1.65	1.61	1.37	1.31	1.34	1.11
ME-3	1.24	1.12	1.31	1.17	1.70	1.29	1.10	1.31	1.36	1.26	0.76
ME-4	1.25	1.27	1.13	1.54	1.06	1.34	1.06	1.41	1.17	1.35	0.98
ME-5	1.29	1.34	1.42	1.39	1.48	1.42	1.18	1.13	1.27	1.18	1.08
ME-6	1.17	1.08	1.53	1.27	1.15	1.20	1.21	1.18	1.04	1.07	1.02
ME-7	1.07	0.95	1.21	1.26	1.09	1.18	1.11	1.24	0.62	1.32	0.76
ME-8	1.10	1.09	1.05	1.37	1.20	1.27	0.98	1.18	1.02	1.01	0.94
ME-9	0.95	0.98	0.88	1.02	1.14	1.07	1.23	0.94	0.82	0.88	0.59
Large-ME	0.89	1.01	0.93	1.10	0.94	0.93	0.89	1.03	0.71	0.74	0.56

Table 1: Avg. Returns and then β: Stocks sorted on ME (down) then pre-ranking βs (across), July 1963 to Dec 1990

	1A	1B	2	3	4	5	6	7	8	9	10A	10B
						Panel A: Portfolios Formed on Size						
Return	1.64	1.16	1.29	1.24	1.25	1.29	1.17	1.07	1.10	0.95	0.88	0.90
β	1.44	1.44	1.39	1.34	1.33	1.24	1.22	1.16	1.08	1.02	0.95	0.90
ln(ME)	1.98	3.18	3.63	4.10	4.50	4.89	5.30	5.73	6.24	6.82	7.39	8.44
ln(BE/ME)	−0.01	−0.21	−0.23	−0.26	−0.32	−0.36	−0.36	−0.44	−0.40	−0.42	−0.51	−0.65
ln(A/ME)	0.73	0.50	0.46	0.43	0.37	0.32	0.32	0.24	0.29	0.27	0.17	−0.03
ln(A/BE)	0.75	0.71	0.69	0.69	0.68	0.67	0.68	0.67	0.69	0.70	0.68	0.62
E/P dummy	0.26	0.14	0.11	0.09	0.06	0.04	0.04	0.03	0.03	0.02	0.02	0.01
E(+)/P	0.09	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.09	0.09
Firms	772	189	236	170	144	140	128	125	119	114	60	64

	1A	1B	2	3	4	5	6	7	8	9	10A	10B
						Panel B: Portfolios Formed on Pre-Ranking β						
Return	1.20	1.20	1.32	1.26	1.31	1.30	1.30	1.23	1.23	1.33	1.34	1.18
β	0.81	0.79	0.92	1.04	1.13	1.19	1.26	1.32	1.41	1.52	1.63	1.73
ln(ME)	4.21	4.86	4.75	4.68	4.59	4.48	4.36	4.25	3.97	3.78	3.52	3.15
ln(BE/ME)	−0.18	−0.13	−0.22	−0.21	−0.23	−0.22	−0.22	−0.25	−0.23	−0.27	−0.31	−0.50
ln(A/ME)	0.60	0.66	0.49	0.45	0.42	0.42	0.45	0.42	0.47	0.46	0.46	0.31
ln(A/BE)	0.78	0.79	0.71	0.66	0.64	0.65	0.67	0.67	0.70	0.73	0.77	0.81
E/P dummy	0.12	0.06	0.09	0.09	0.08	0.09	0.10	0.12	0.12	0.14	0.17	0.23
E(+)/P	0.11	0.12	0.10	0.10	0.10	0.10	0.10	0.09	0.10	0.09	0.09	0.08
Firms	116	80	185	181	179	182	185	205	227	267	165	291

Table 2: Properties of portfolios formed on size and pre-ranking βs,
July 1963 to December 1990

β	ln(ME)	ln(BE/ME)	ln(A/ME)	ln(A/BE)	E/P Dummy	E(+)/P
0.15 (0.46)						
	−0.15 (−2.58)					
−0.37 (−1.21)	−0.17 (−3.41)					
		0.50 (5.71)				
			0.50 (5.69)	−0.57 (−5.34)		
					0.57 (2.28)	4.72 (4.57)
	−0.11 (−1.99)	0.35 (4.44)				
	−0.11 (−2.06)		0.35 (4.32)	−0.50 (−4.56)		
	−0.16 (−3.06)				0.06 (0.38)	2.99 (3.04)
	−0.13 (−2.47)	0.33 (4.46)			−0.14 (−0.90)	0.87 (1.23)
	−0.13 (−2.47)		0.32 (4.28)	−0.46 (−4.45)	−0.08 (−0.56)	1.15 (1.57)

Table 3: Avg. slope coefficients (and t-statistics) from Fama-French month-by-month regressions of stock returns on β, size, BE/ME, Leverage, and E/P

Fama and Macbeth

Statistic	1	2	3	4	5	6	7	8	9	10
	Portfolios for Estimation Period 1934–38									
$\hat{\beta}_{p,t-1}$	.322	.508	.651	.674	.695	.792	.921	.942	.970	1.005
$s(\hat{\beta}_{p,t-1})$	.027	.027	.025	.023	.028	.026	.032	.029	.034	.027
$r(R_p, R_m)^2$	.709	.861	.921	.936	.912	.941	.932	.946	.933	.958
$s(R_p)$	.040	.058	.072	.074	.077	.087	.101	.103	.106	.109
$s(\hat{\epsilon}_p)$	.022	.022	.020	.019	.023	.021	.026	.024	.028	.022
$\bar{s}_{p,t-1}(\hat{\epsilon}_i)$	.085	.075	.083	.078	.090	.095	.109	.106	.111	.097
$s(\hat{\epsilon}_p)/\bar{s}_{p,t-1}(\hat{\epsilon}_i)$	.259	.293	.241	.244	.256	.221	.238	.226	.252	.227
	Portfolios for Estimation Period 1942–46									
$\hat{\beta}_{p,t-1}$	.467	.537	.593	.628	.707	.721	.770	.792	.805	.894
$s(\hat{\beta}_{p,t-1})$	.045	.041	.044	.037	.027	.032	.035	.035	.028	.040
$r(R_p, R_m)^2$	.645	.745	.753	.829	.919	.898	.889	.898	.934	.896
$s(R_p)$	.035	.037	.041	.041	.044	.046	.049	.050	.050	.057
$s(\hat{\epsilon}_p)$	.021	.019	.020	.017	.013	.015	.016	.016	.013	.018
$\bar{s}_{p,t-1}(\hat{\epsilon}_i)$	.055	.055	.063	.058	.058	.063	.064	.064	.062	.069
$s(\hat{\epsilon}_p)/\bar{s}_{p,t-1}(\hat{\epsilon}_i)$	.382	.345	.317	.293	.224	.238	.250	.250	.210	.261
	Portfolios for Estimation Period 1950–54									
$\hat{\beta}_{p,t-1}$	.418	.590	.694	.751	.777	.784	.929	.950	.996	1.014
$s(\hat{\beta}_{p,t-1})$	.042	.047	.045	.037	.038	.035	.050	.038	.035	.029
$r(R_p, R_m)^2$	.629	.723	.798	.872	.878	.895	.856	.913	.933	.954
$s(R_p)$	.019	.025	.028	.029	.030	.030	.036	.036	.037	.038
$s(\hat{\epsilon}_p)$	.012	.013	.013	.010	.010	.010	.014	.011	.010	.008
$\bar{s}_{p,t-1}(\hat{\epsilon}_i)$	.040	.044	.046	.048	.051	.051	.052	.053	.054	.057
$s(\hat{\epsilon}_p)/\bar{s}_{p,t-1}(\hat{\epsilon}_i)$	.300	.295	.283	.208	.196	.196	.269	.208	.185	.140
	Portfolios for Estimation Period 1958–62									
$\hat{\beta}_{p,t-1}$	.626	.635	.719	.801	.817	.860	.920	.950	.975	.995
$s(\hat{\beta}_{p,t-1})$	.043	.048	.039	.046	.047	.033	.037	.038	.032	.037
$r(R_p, R_m)^2$	.783	.745	.851	.835	.838	.920	.913	.915	.939	.925
$s(R_p)$	.030	.031	.033	.037	.038	.038	.041	.042	.043	.044
$s(\hat{\epsilon}_p)$	.014	.016	.013	.015	.015	.011	.012	.012	.011	.012
$\bar{s}_{p,t-1}(\hat{\epsilon}_i)$	.049	.052	.056	.059	.064	.061	.070	.069	.068	.064
$s(\hat{\epsilon}_p)/\bar{s}_{p,t-1}(\hat{\epsilon}_i)$	.286	.308	.232	.254	.234	.180	.171	.174	.162	.188

Table 4: Summary statistics for different subperiods

Statistic	11	12	13	14	15	16	17	18	19	20
	Portfolios for Estimation Period 1934–38									
$\hat{\beta}_{p.t-1}$	1.046	1.122	1.181	1.192	1.196	1.295	1.335	1.396	1.445	1.458
$s(\hat{\beta}_{p.t-1})$	.028	.031	.035	.028	.029	.032	.032	.053	.039	.053
$r(R_p, R_m)^2$	.959	.956	.951	.969	.966	.966	.967	.922	.958	.927
$s(R_p)$	.113	.122	.128	.128	.129	.140	.144	.154	.156	.160
$s(\hat{\epsilon}_p)$	.023	.026	.029	.023	.024	.026	.026	.043	.032	.043
$\bar{s}_{p.t-1}(\hat{\epsilon}_i)$	.094	.124	.120	.122	.132	.125	.129	.158	.145	.170
$s(\hat{\epsilon}_p)/\bar{s}_{p,t-1}(\hat{\epsilon}_i)$	.245	.210	.242	.188	.182	.208	.202	.272	.221	.253
	Portfolios for Estimation Period 1942–46									
$\hat{\beta}_{p.t-1}$	.949	.952	1.010	1.038	1.254	1.312	1.316	1.473	1.631	1.661
$s(\hat{\beta}_{p.t-1})$	.031	.036	.040	.030	.034	.039	.041	.084	.083	.077
$r(R_p, R_m)^2$	.942	.923	.917	.954	.958	.951	.945	.839	.867	.887
$s(R_p)$	.059	.060	.063	.064	.077	.081	.081	.097	.105	.106
$s(\hat{\epsilon}_p)$	.014	.016	.018	.014	.016	.018	.019	.039	.038	.036
$\bar{s}_{p.t-1}(\hat{\epsilon}_i)$	.073	.074	.085	.077	.096	.083	.086	.134	.117	.122
$s(\hat{\epsilon}_p)/\bar{s}_{p,t-1}(\hat{\epsilon}_i)$	.192	.216	.212	.182	.167	.217	.221	.291	.325	.295
	Portfolios for Estimation Period 1950–54									
$\hat{\beta}_{p.t-1}$	1.117	1.123	1.131	1.134	1.186	1.235	1.295	1.324	1.478	1.527
$s(\hat{\beta}_{p.t-1})$	.039	.027	.044	.033	.037	.049	.045	.046	.058	.086
$r(R_p, R_m)^2$	.934	.968	.919	.952	.944	.915	.933	.934	.917	.841
$s(R_p)$	.042	.041	.043	.042	.044	.047	.049	.050	.056	.060
$s(\hat{\epsilon}_p)$	.011	.007	.012	.009	.010	.014	.013	.013	.016	.024
$\bar{s}_{p.t-1}(\hat{\epsilon}_i)$	.066	.057	.066	.060	.064	.064	.065	.068	.076	.088
$s(\hat{\epsilon}_p)/\bar{s}_{p.t-1}(\hat{\epsilon}_i)$	.167	.123	.182	.150	.156	.219	.200	.192	.210	.273
	Portfolios for Estimation Period 1958–62									
$\hat{\beta}_{p.t-1}$	1.013	1.019	1.037	1.048	1.069	1.081	1.092	1.098	1.269	1.388
$s(\hat{\beta}_{p.t-1})$	.038	.031	.036	.033	.036	.038	.045	.045	.048	.065
$r(R_p, R_m)^2$	.922	.948	.934	.945	.936	.931	.907	.910	.922	.886
$s(R_p)$	.045	.045	.046	.046	.047	.048	.049	.049	.056	.063
$s(\hat{\epsilon}_p)$	.013	.010	.012	.011	.012	.013	.015	.015	.016	.021
$\bar{s}_{p.t-1}(\hat{\epsilon}_i)$	.069	.066	.067	.062	.070	.072	.076	.068	.070	.078
$s(\hat{\epsilon}_p)/\bar{s}_{p.t-1}(\hat{\epsilon}_i)$	.188	.152	.179	.177	.171	.180	.197	.220	.228	.269

Table 4 (continued)

Table 5

SUMMARY RESULTS FOR THE REGRESSION

$$R_p = \hat{\gamma}_{0t} + \hat{\gamma}_{1t}\hat{\beta}_p + \hat{\gamma}_{2t}\hat{\beta}^2{}_p + \hat{\gamma}_{3t}\bar{s}_p(\hat{\epsilon}_i) + \hat{\eta}_{pt}$$

PERIOD	STATISTIC																			
	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\overline{\hat{\gamma}_0 - R_f}$	$s(\hat{\gamma}_0)$	$s(\hat{\gamma}_1)$	$s(\hat{\gamma}_2)$	$s(\hat{\gamma}_3)$	$\rho_0(\hat{\gamma}_0 - R_f)$	$\rho_M(\hat{\gamma}_1)$	$\rho_0(\hat{\gamma}_2)$	$\rho_0(\hat{\gamma}_3)$	$t(\bar{\hat{\gamma}}_0)$	$t(\bar{\hat{\gamma}}_1)$	$t(\bar{\hat{\gamma}}_2)$	$t(\bar{\hat{\gamma}}_3)$	$t(\overline{\hat{\gamma}_0 - R_f})$	$\bar{r}^2$	$s(r^2)$
Panel A:																				
1935–6/68 ..	.0061	.0085	...	...	.0048	.038	.066	...	...	.15	.02	...	...	3.24	2.57	...	...	2.55	.29	.30
1935–45	.0039	.0163	...	...	.0037	.052	.098	...	...	.10	−.03	...	...	.86	1.92	...	...	.82	.29	.29
1946–55	.0087	.0027	...	...	.0078	.026	.041	...	...	.18	.07	...	...	3.71	.70	...	...	3.31	.31	.32
1956–6/68 ..	.0060	.0062	...	...	.0034	.030	.044	...	...	.27	.15	...	...	2.45	1.73	...	...	1.39	.28	.29
1935–40	.0024	.0109	...	...	.0023	.064	.116	...	...	.07	−.09	...	...	.32	.79	...	...	.31	.23	.30
1941–45	.0056	.0229	...	...	.0054	.034	.069	...	...	.23	.15	...	...	1.27	2.55	...	...	1.22	.37	.28
1946–50	.0050	.0029	...	...	.0044	.031	.047	...	...	.20	.04	...	...	1.27	.48	...	...	1.10	.39	.33
1951–55	.0123	.0024	...	...	.0111	.019	.035	...	...	.20	.08	...	...	5.06	.53	...	...	4.56	.24	.29
1956–60	.0148	−.0059	...	...	.0128	.020	.034	...	...	.37	.18	...	...	5.68	−1.37	...	...	4.89	.22	.31
1961–6/68 ..	.0001	.0143	...	...	−.0029	.034	.048	...	...	.22	.09	...	...	.03	2.81	...	...	−.80	.32	.27
Panel B:																				
1935–6/68 ..	.0049	.0105	−.0008	...	.0036	.052	.118	.056	...	.03	−.11	−.11	...	1.92	1.79	−.29	...	1.42	.32	.31
1935–45	.0074	.0079	.0040	...	.0073	.061	.139	.074	...	−.10	−.31	−.21	...	1.39	.65	.61	...	1.36	.32	.30
1946–55	−.0002	.0217	−.0087	...	−.0012	.036	.095	.034	...	.04	.00	.00	...	−.07	2.51	−2.83	...	−.38	.36	.32
1956–6/68 ..	.0069	.0040	.0013	...	.0043	.054	.116	.053	...	.17	.07	.03	...	1.56	.42	.29	...	.97	.30	.30
1935–40	.0013	.0141	−.0017	...	.0012	.069	.160	.075	...	−.13	−.36	−.35	...	.16	.75	−.19	...	.14	.24	.30
1941–45	.0148	.0004	.0108	...	.0146	.050	.111	.073	...	−.04	−.19	−.04	...	2.28	.03	1.15	...	2.24	.39	.29
1946–50	−.0008	.0152	−.0051	...	−.0015	.037	.104	.032	...	.14	.04	.00	...	−.18	1.14	−1.24	...	−.32	.44	.32
1951–55	.0004	.0281	−.0122	...	−.0008	.030	.085	.035	...	−.17	−.14	−.01	...	.10	2.55	−2.72	...	−.20	.28	.29
1956–60	.0128	−.0015	−.0020	...	.0108	.030	.072	.029	...	.35	.11	.26	...	3.38	−.16	−.54	...	2.84	.25	.31
1961–6/68 ..	.0029	.0077	.0034	...	−.0000	.066	.138	.064	...	.14	.06	−.01	...	.42	.53	.51	...	−.01	.34	.29

Table 5 (continued)

Period	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\overline{\hat{\gamma}_0 - R_f}$	$s(\hat{\gamma}_0)$	$s(\hat{\gamma}_1)$	$s(\hat{\gamma}_2)$	$s(\hat{\gamma}_3)$	$\rho_0(\hat{\gamma}_0 - R_f)$	$\rho_M(\hat{\gamma}_1)$	$\rho_0(\hat{\gamma}_2)$	$\rho_0(\hat{\gamma}_3)$	$t(\bar{\hat{\gamma}}_0)$	$t(\bar{\hat{\gamma}}_1)$	$t(\bar{\hat{\gamma}}_2)$	$t(\bar{\hat{\gamma}}_3)$	$t(\overline{\hat{\gamma}_0 - R_f})$	$\bar{r}^2$	$s(r^2)$
Panel C:																				
1935–6/68 ..	.0054	.0072	...	.0198	.0041	.052	.065	...	.868	.04	—.12	...	—.04	2.10	2.20	...	.46	1.59	.32	.31
1935–45	.0017	.0104	...	.0841	.0015	.073	.083	...	.921	—.00	—.26	...	—.08	.26	1.41	...	1.05	.24	.32	.31
1946–55	.0110	.0075	...	—.1052	.0100	.032	.056	...	.609	.08	.02	...	—.20	3.78	1.47	...	—1.89	3.46	.34	.32
1956–6/68 ..	.0042	.0041	...	.0633	.0016	.040	.052	...	.984	.12	.08	...	.03	1.28	.96	...	.79	.50	.30	.29
1935–40	.0036	.0119	...	—.0170	.0035	.082	.105	...	.744	—.03	—.26	...	—.18	.37	.97	...	—.19	.36	.25	.30
1941–45	—.0006	.0085	...	.2053	—.0009	.061	.052	...	1.091	.07	—.29	...	—.02	—.08	1.25	...	1.46	—.11	.41	.30
1946–50	.0069	.0081	...	—.0920	.0062	.034	.066	...	.504	.14	.06	...	—.02	1.56	.95	...	—1.41	1.40	.42	.33
1951–55	.0150	.0069	...	—.1185	.0138	.029	.043	...	.702	.06	—.18	...	—.32	4.05	1.24	...	—1.31	3.72	.27	.29
1956–60	.0127	—.0081	...	.0728	.0107	.037	.045	...	1.164	.15	.15	...	.21	2.68	—1.40	...	.48	2.26	.26	.30
1961–6/68 ..	—.0014	.0122	...	.0570	—.0044	.042	.055	...	.850	.10	.00	...	—.19	—.32	2.12	...	.64	—.98	.33	.27
Panel D:																				
1935–6/68 ..	.0020	.0114	—.0026	.0516	.0008	.075	.123	.060	.929	—.09	—.09	—.12	—.10	.55	1.85	—.86	1.11	.20	.34	.31
1935–45	.0011	.0118	—.0009	.0817	.0010	.103	.146	.079	1.003	—.20	—.23	—.24	—.15	.13	.94	—.14	.94	.11	.34	.31
1946–55	.0017	.0209	—.0076	—.0378	.0008	.042	.096	.038	.619	—.10	—.00	—.01	—.20	.44	2.39	—2.16	—.67	.20	.36	.32
1956–6/68 ..	.0031	.0034	—.0000	.0966	.0005	.065	.122	.055	1.061	.12	.03	.01	—.05	.59	.34	—.00	1.11	.10	.32	.29
1935–40	.0009	.0156	—.0029	.0025	.0008	.112	.171	.085	.826	—.16	—.23	—.26	—.12	.07	.78	—.29	.03	.06	.26	.30
1941–45	.0015	.0073	.0014	.1767	.0012	.092	.109	.072	1.181	—.28	—.21	—.22	—.18	.12	.52	.15	1.16	.10	.43	.31
1946–50	.0011	.0141	—.0040	—.0313	.0004	.047	.106	.042	.590	—.10	.03	—.01	—.12	.18	1.03	—.73	—.41	.07	.44	.33
1951–55	.0023	.0277	—.0112	—.0443	.0011	.037	.085	.034	.651	—.11	—.13	—.01	—.28	.48	2.53	—2.54	—.53	.23	.29	.30
1956–60	.0103	—.0047	—.0020	.0979	.0083	.049	.078	.032	1.286	—.16	.19	—.01	.02	1.63	—.47	—.49	.59	1.31	.28	.30
1961–6/68 ..	—.0017	.0088	.0013	.0957	—.0046	.073	.144	.066	.887	.20	.00	.01	—.15	—.21	.58	.19	1.02	—.60	.35	.29

Black, Scholes and Jensen

Item*	Portfolio Number 1	2	3	4	5	6	7	8	9	10	$\overline{R}_M$
$\hat{\beta}$	1.5614	1.3838	1.2483	1.1625	1.0572	0.9229	0.8531	0.7534	0.6291	0.4992	1.0000
$\hat{\alpha} \bullet 10^2$	-0.0829	-0.1938	-0.0649	-0.0167	-0.0543	0.0593	0.0462	0.0812	0.1968	0.2012	
$t(\hat{\alpha})$	-0.4274	-1.9935	-0.7597	-0.2468	-0.8869	0.7878	0.7050	1.1837	2.3126	1.8684	
$r(\tilde{R}, \tilde{R}_M)$	0.9625	0.9875	0.9882	0.9914	0.9915	0.9833	0.9851	0.9793	0.9560	0.8981	
$r(\tilde{e}_t, \tilde{e}_{t-1})$	0.0549	-0.0638	0.0366	0.0073	-0.0708	-0.1248	0.1294	0.1041	0.0444	0.0992	
$\sigma(\tilde{e})$	0.0393	0.0197	0.0173	0.0137	0.0124	0.0152	0.0133	0.0139	0.0172	0.0218	
$\overline{R}$	0.0213	0.0177	0.0171	0.0163	0.0145	0.0137	0.0126	0.0115	0.0109	0.0091	0.0142
σ	0.1445	0.1248	0.1126	0.1045	0.0950	0.0836	0.0772	0.0685	0.0586	0.0495	0.0891

* $\overline{R}_M$ = average monthly excess returns, σ = standard deviation of the monthly excess returns, r = correlation coefficient.

Table 6: Summary statistics for time-series tests, January 1931 to December 1965

Summary of Coefficients for the Subperiods

Item*	Subperiod†	Portfolio Number										
		1	2	3	4	5	6	7	8	9	10	M_M
$\hat{\beta}$	1	1.5416	1.3993	1.2620	1.1813	1.0750	0.9197	0.8569	0.7510	0.6222	0.4843	1.0000
	2	1.7157	1.3196	1.1938	1.0861	0.9697	0.9254	0.8114	0.7675	0.6647	0.5626	1.0000
	3	1.5427	1.3598	1.1822	1.1216	1.0474	0.9851	0.9180	0.7714	0.6547	0.4868	1.0000
	4	1.4423	1.2764	1.1818	1.0655	0.9957	0.9248	0.8601	0.7800	0.6614	0.6226	1.0000
$\hat{\alpha} \bullet 10^2$	1	0.7366	0.1902	0.3978	0.1314	-0.0650	-0.0501	-0.2190	-0.3786	-0.2128	-0.0710	
	2	-0.2197	-0.1300	-0.1224	0.0653	-0.0805	0.0914	0.1306	0.0760	0.2685	0.1478	
	3	-0.4614	-0.3994	-0.1189	0.0052	0.0002	-0.0070	0.1266	0.2428	0.3032	0.2035	
	4	-0.4475	-0.2536	-0.2329	-0.0654	0.0840	0.1356	0.1218	0.3257	0.3338	0.3685	
$t(\hat{\alpha})$	1	1.3881	0.6121	1.4037	0.6484	-0.3687	-0.1882	-1.0341	-1.7601	-0.7882	-0.1978	
	2	-0.4256	-0.7605	-0.8719	0.5019	-0.6288	0.8988	1.1377	0.6178	1.7853	0.8377	
	3	-2.9030	-3.6760	-1.5160	0.0742	0.0029	-0.1010	1.8261	3.3768	3.3939	1.9879	
	4	-2.8761	-2.4603	-2.7886	-0.7722	1.1016	1.7937	1.6769	3.8772	3.0651	3.2439	
$\bar{R}$	1	0.0412	0.0326	0.0317	0.0272	0.0230	0.0197	0.0166	0.0127	0.0115	0.0099	0.0220
	2	0.0233	0.0183	0.0165	0.0168	0.0136	0.0147	0.0134	0.0122	0.0126	0.0098	0.0149
	3	0.0126	0.0112	0.0120	0.0126	0.0117	0.0109	0.0115	0.0110	0.0103	0.0075	0.0112
	4	0.0082	0.0082	0.0081	0.0087	0.0096	0.0095	0.0088	0.0101	0.0092	0.0092	0.0088
σ	1	0.2504	0.2243	0.2023	0.1886	0.1715	0.1484	0.1377	0.1211	0.1024	0.0850	0.1587
	2	0.1187	0.0841	0.0758	0.0690	0.0618	0.0586	0.0519	0.0494	0.0441	0.0392	0.0624
	3	0.0581	0.0505	0.0436	0.0413	0.0385	0.0364	0.0340	0.0289	0.0253	0.0203	0.0363
	4	0.0577	0.0503	0.0463	0.0420	0.0391	0.0365	0.0340	0.0312	0.0277	0.0265	0.0386

* $\bar{R}_M$ = average monthly excess returns, σ = standard deviation of the monthly excess returns.

†Subperiod 1 = January, 1931-September, 1939; 2 = October, 1939-June, 1948; 3 =July, 1948-March, 1957; 4 = April, 1957-December, 1965.

Table 7: Summary of coefficients from time-series analysis

Chung, Johnson and Schill

	Daily	Weekly	Monthly	Quarterly	Semiannual
	A. Size-Sorted Portfolios				
Number of portfolio-period observations	428,050	88,700	41,393	13,798	6,899
Mean	.0006	.0022	.0103	.0335	.0641
Variance	.0001	.0005	.0072	.0412	.0706
Skewness	−.5905	−.3963	2.318	5.059	2.5872
Kurtosis	11.970	4.541	25.66	58.86	20.25
Jarque-Bera statistic	1,457,597.5**	11,098.2	922,517**	1,852,337**	93,228.4**
Kolmogorov statistic	.0591**	.0459**	.0980**	.1347**	.0914**
	B. Book-to-Market-Ratio-Sorted Portfolios				
Number of portfolio-period observations	353,750	75,700	17,400	5,800	2,900
Mean	.00070	.0022	.0074	.0228	.0466
Variance	.00007	.0006	.0044	.0214	.0529
Skewness	−.7913	−.3951	−.0124	.3708	.7880
Kurtosis	13.388	5.120	3.218	1.1643	1.7794
Jarque-Bera statistic	1,627,167.3**	16,142.5	35.06	947.3	480.1
Kolmogorov statistic	.0604**	.0499**	.0443**	.0471**	.0624**

Table 8: Summary Statistics of Portfolio Returns

	Systematic Comoments							
	2nd to 3rd	2nd to 4th	2nd to 5th	2nd to 6th	2nd to 7th	2nd to 8th	2nd to 9th	2nd to 10th
	A. Size-Sorted Portfolios							
Daily (Periods = 8,561):								
s	.0006**	.0007**	.0007**	.0008**	.0008**	.0007**	.0005*	.0007**
	(7.14)	(6.96)	(6.79)	(6.23)	(5.27)	(3.61)	(2.33)	(3.13)
h	−.0001	−.0003*	−.0003*	−.0004*	−.0004	−.0004	−.0005*	−.0003
	(−1.00)	(−1.98)	(−1.86)	(−1.96)	(−1.49)	(−1.56)	(−1.98)	(−1.07)
Adjusted R^2 [joint test]	.200	.206	.215	.220	.226	.233	.236	.237
	[3.77*]	[1.96]	[1.79]	[1.64]	[1.54]	[1.53]	[1.50]	[1.47]
Weekly (Periods = 1,774):								
s	.003**	.002**	.002**	.002**	.002**	.002**	.002**	.001
	(8.06)	(4.75)	(5.07)	(4.78)	(4.59)	(3.03)	(2.50)	(1.48)
h	−.001**	−.002**	−.002**	−.002*	−.001	−.001	−.001	-.000
	(−2.67)	(−3.63)	(−3.63)	(−1.99)	(−1.49)	(−.78)	(−.93)	(−.26)
Adjusted R^2 [joint test]	.298	.304	.312	.317	.321	.327	.331	.335
	[69.0**]	[30.8**]	[33.0**]	[22.7**]	[19.3**]	[8.92*]	[6.59*]	[2.85]
Monthly (Periods = 828):								
s	.005**	.003*	−.001	.001	−.001	−.001	.001	.003
	(3.83)	(1.89)	(−.38)	(.26)	(−.25)	(−.43)	(.39)	(.69)
h	−.002*	−.003*	−.002	−.002	−.001	−.001	−.001	−.002
	(−1.70)	(−1.89)	(−.86)	(−.53)	(−.39)	(−.33)	(−.18)	(−.30)
Adjusted R^2 [joint test]	.361	.367	.376	.379	.385	.390	.391	.394
	[19.0**]	[3.23]	[.78]	[.44]	[1.19]	[3.27]	[.10]	[.37]
Quarterly (Periods = 276):								
s	.002	.005	−.000	−.003	−.011	−.008	−.016	−.029
	(.24)	(.69)	(−.05)	(−.30)	(−.90)	(−.58)	(−1.15)	(−1.81)
h	−.005	−.004	.003	.017	.007	.007	.020	.029
	(−.83)	(−.51)	(.39)	(2.19)	(.65)	(.57)	(1.53)	(2.09)
Adjusted R^2 [joint test]	.463	.471	.482	.488	.494	.496	.496	.488
	[2.06]	[4.04]	[2.46]	[1.50]	[.51]	[.11]	[1.03]	[3.13]
Semiannual (Periods = 138):								
s	−.011	.003	−.030	−.0003	−.058	−.033	.013	.049
	(−1.30)	(.31)	(−2.19)	(−.01)	(−1.23)	(−.62)	(.22)	(.65)
h	−.012	−.021*	−.026*	.007	−.046	−.037	−.071	−.204*
	(−1.48)	(−2.28)	(−1.92)	(.28)	(−.87)	(−.61)	(−1.12)	(−1.68)
Adjusted R^2 [joint test]	.481	.488	.503	.508	.510	.507	.499	.489
	[3.45]	[3.64]	[8.05*]	[1.97]	[5.88]	[2.26]	[2.63]	[2.36]
	B. Book-to-Market-Sorted Portfolios							
Daily (Periods = 8,561):								
s	.0009**	.0010**	.0009**	.0010**	.0004*	.0006*	.0006*	.0008**
	(5.27)	(5.65)	(4.66)	(4.84)	(1.94)	(2.28)	(2.30)	(2.72)
h	−.001**	−.001**	−.001**	−.001**	−.001**	−.001**	−.001**	−.0005
	(−7.26)	(−6.68)	(−5.27)	(−2.88)	(−3.33)	(−3.47)	(−2.38)	(−1.57)
Adjusted R^2 [joint test]	.133	.136	.140	.142	.146	.149	.148	.143
	[1.93]	[1.73]	[1.64]	[1.58]	[1.45]	[1.36]	[1.35]	[1.29]
Weekly (Periods = 1,774):								
s	.002**	.002**	.001*	.001	.001	.0005	.0002	.0004
	(4.09)	(2.78)	(1.71)	(1.62)	(1.51)	(.56)	(.23)	(.41)
h	−.003**	−.003**	−.002**	−.002*	.0002	−.0001	.002	.0019
	(−7.57)	(−5.91)	(−3.43)	(−1.99)	(.20)	(−.08)	(1.37)	(1.41)
Adjusted R^2 [joint test]	.239	.247	.252	.256	.261	.268	.270	.270
	[72.7**]	[33.4**]	[13.7**]	[5.05]	[1.04]	[.17]	[1.54]	[.03]
Monthly (Periods = 828):								
s	.006**	.004	−.002	−.003	−.004	−.004	−.004	−.001
	(2.83)	(1.53)	(−.65)	(−.94)	(−1.33)	(−.84)	(−.83)	(−.23)
h	−.011**	−.008**	−.007**	−.001	−.001	−.001	.005	.001
	(−6.15)	(−3.97)	(−3.13)	(−.44)	(−.21)	(−.16)	(.94)	(.20)
Adjusted R^2 [joint test]	.440	.447	.453	.460	.464	.469	.468	.467
	[50.0**]	[16.3**]	[17.2**]	[5.54]	[7.32*]	[6.04*]	[.69]	[1.11]
Quarterly (Periods = 276):								
s	−.013	−.008	−.020	−.024	−.038	.0008	−.037	−.040
	(−2.56)	(−1.09)	(−2.68)	(−1.66)	(−2.46)	(.04)	(−1.47)	(−1.52)
h	−.024**	−.024**	−.009	.0004	.008	−.017	.022	.025
	(−4.38)	(−3.76)	(−1.29)	(.32)	(.63)	(−.98)	(1.21)	(1.17)
Adjusted R^2 [joint test]	.594	.601	.617	.625	.631	.635	.642	.637
	[31.7**]	[23.4**]	[19.0**]	[9.40**]	[13.9**]	[2.55]	[2.49]	[1.18]
Semiannual (Periods = 138):								
s	−.018	−.017	−.055	−.121	−.136	−.203	−.325	−.312
	(−2.07)	(−.81)	(−1.92)	(−1.70)	(−2.16)	(−2.26)	(−2.17)	(−.203)
h	−.013	−.014	−.017	.015	.017	.053	.111	.078
	(−1.01)	(−.92)	(−.95)	(.20)	(.34)	(−.66)	(.96)	(.65)
Adjusted R^2 [joint test]	.666	.676	.687	.701	.707	.711	.706	.704
	[6.44*]	[2.62]	[5.15]	[4.18]	[9.29**]	[4.07]	[5.43]	[10.2**]

Table 9: Fama-Macbeth regression results for Fama-French factors and systematic co-moments, * one-tail significant at 5% level, **one-tail significant at 1% level

Return Frequency	Size-Sorted Portfolios: s	h	Mean Adjusted R^2 [Joint Test]	Book-to-Market-Ratio-Sorted Portfolios: s	h	Mean Adjusted R^2 [Joint Test]
Daily	.0005**	−.0005**	.254	−.0001	−.0009**	.191
	(4.23)	(−2.74)	[1.69]	(−.57)	(−5.33)	[31.5**]
Weekly	.0020**	−.0026**	.351	−.0009	−.0046**	.314
	(4.18)	(−4.70)	[31.5**]	(−1.01)	(−8.39)	[83.5**]
Monthly	.0075**	−.0034*	.436	.0004	−.0145**	.508
	(4.56)	(−1.97)	[18.5**]	(.13)	(−7.98)	[45.8**]
Quarterly	.0219**	.0044	.534	.0030	−.0317**	.676
	(2.48)	(.43)	[7.59*]	(.38)	(−6.25)	[18.4**]
Semiannual	.0245**	−.0197**	.544	−.0224	−.0407**	.755
	(2.50)	(−2.68)	[14.4**]	(−1.23)	(−4.01)	[16.6**]

NOTE.—The sample consists of daily, weekly, monthly, quarterly, and semiannual returns of 50 equal-sized portfolios. As denoted, the portfolios are sorted in two ways. The size-sorted portfolios are sorted by beginning-of-period size and contain all CRSP-listed ordinary common equities from 1930 to 1998 for the monthly, quarterly, and semiannual return intervals and from 1965 to 1998 for the daily and weekly return intervals. The book-to-market-ratio-sorted portfolios are sorted by beginning of period book-to-market ratio and contain all CRSP- and Compustat-listed ordinary common equities from 1970 to 1998 for all return intervals. The Fama-French factor loadings s and h are computed from the three-factor model estimates using portfolio returns over the past five years. The standard moments are estimates using the same rolling five-year portfolio return data. For each period the portfolio return is regressed on the Fama-French factor loadings, beta, and moments 3–10. The mean coefficient estimates across the sample period are reported. The joint test is a Wald test of the joint significance of the s and h estimates.

* One-tail significance at the 5% level.

** One-tail significance at the 1% level.

Table 10: Fama-Macbeth regression results for Fama-French factors and standard moments of order 3 to 10

Javier Estrada

Summary statistics (monthly stock returns)

Market	MR	σ	ρ	β	Σ	Θ	β^{D}	SSkw	Start
Australia	0.86	5.63	0.57	0.77	3.94	0.70	0.89	0.24	Jan/88
Austria	0.54	6.92	0.38	0.63	4.81	0.63	0.98	0.67	Jan/88
Belgium	1.05	4.94	0.58	0.69	3.44	0.70	0.78	1.46	Jan/88
Canada	0.83	5.15	0.72	0.89	3.86	0.79	0.98	−3.28	Jan/88
Denmark	1.17	5.37	0.59	0.76	3.81	0.72	0.89	0.13	Jan/88
Finland	1.65	9.74	0.55	1.29	6.66	0.67	1.43	0.95	Jan/88
France	1.13	5.66	0.69	0.94	4.01	0.79	1.02	0.12	Jan/88
Germany	1.00	6.04	0.65	0.95	4.46	0.79	1.14	−1.57	Jan/88
Greece	1.55	11.34	0.27	0.72	6.67	0.60	1.28	8.63	Jan/88
Hong Kong	1.44	8.45	0.59	1.19	5.80	0.67	1.26	1.37	Jan/88
Ireland	0.99	5.69	0.66	0.90	3.98	0.75	0.96	0.53	Jan/88
Italy	0.72	7.06	0.52	0.88	4.79	0.67	1.04	1.41	Jan/88
Japan	−0.01	7.06	0.76	1.29	4.71	0.80	1.21	2.14	Jan/88
Netherlands	1.18	4.50	0.76	0.82	3.42	0.82	0.90	−3.20	Jan/88
New Zealand	0.35	7.08	0.49	0.84	4.86	0.68	1.06	1.59	Jan/88
Norway	0.88	6.74	0.59	0.95	4.93	0.71	1.13	−2.22	Jan/88
Portugal	0.43	6.66	0.46	0.74	4.42	0.60	0.86	3.20	Jan/88
Singapore	0.94	8.55	0.64	1.32	6.06	0.73	1.42	0.45	Jan/88
Spain	0.96	6.36	0.70	1.07	4.48	0.83	1.19	−0.33	Jan/88
Sweden	1.39	7.37	0.72	1.27	5.33	0.81	1.40	−1.29	Jan/88
Switzerland	1.17	5.14	0.66	0.81	3.63	0.77	0.90	−0.25	Jan/88
UK	0.89	4.69	0.77	0.87	3.21	0.80	0.83	1.43	Jan/88
USA	1.22	4.09	0.81	0.79	3.04	0.88	0.86	−2.23	Jan/88
Argentina	2.96	18.19	0.15	0.66	10.17	0.56	1.82	10.78	Jan/88
Brazil	2.91	17.37	0.35	1.44	11.55	0.58	2.16	2.51	Jan/88
Chile	1.74	7.56	0.32	0.57	5.27	0.56	0.95	−0.42	Jan/88
China	−0.72	12.72	0.37	1.13	7.92	0.54	1.39	4.27	Jan/93
Colombia	0.29	9.68	0.14	0.32	6.55	0.38	0.81	1.41	Jan/93
Czech Rep.	0.24	9.28	0.30	0.66	6.59	0.69	1.29	0.23	Jan/95
Egypt	0.46	8.69	0.25	0.53	5.18	0.61	0.90	4.94	Jan/95
Hungary	1.68	11.84	0.54	1.53	8.17	0.82	1.91	0.94	Jan/95
India	0.42	8.88	0.26	0.54	6.04	0.56	1.10	1.09	Jan/93
Indonesia	1.26	17.08	0.24	0.97	9.88	0.50	1.60	10.38	Jan/88
Israel	0.76	7.13	0.37	0.63	5.42	0.49	0.87	−2.01	Jan/93
Jordan	0.16	4.45	0.11	0.11	3.11	0.32	0.32	−0.80	Jan/88
Korea	0.93	12.56	0.41	1.25	7.68	0.54	1.34	6.83	Jan/88
Malaysia	0.95	10.09	0.42	1.02	6.87	0.60	1.33	3.16	Jan/88
Mexico	2.40	10.41	0.45	1.12	7.67	0.60	1.47	−2.23	Jan/88
Morocco	0.70	4.95	−0.10	−0.12	3.35	0.41	0.39	1.62	Jan/93
Pakistan	−0.02	12.08	0.17	0.49	7.91	0.39	1.00	1.96	Jan/93
Peru	0.97	9.47	0.33	0.74	6.55	0.56	1.19	0.76	Jan/93
Philippines	0.71	10.36	0.44	1.10	6.94	0.63	1.40	2.78	Jan/88
Poland	2.59	17.86	0.39	1.66	10.03	0.62	2.02	11.00	Jan/93
Russia	3.59	22.22	0.50	2.69	15.27	0.65	2.85	0.56	Jan/95
South Africa	0.78	8.20	0.56	1.10	6.02	0.68	1.33	−1.90	Jan/93
Sri Lanka	0.10	10.44	0.24	0.61	6.67	0.51	1.11	4.16	Jan/93
Taiwan	1.27	12.47	0.29	0.87	8.19	0.57	1.49	2.44	Jan/88
Thailand	0.72	12.73	0.46	1.41	8.80	0.62	1.75	1.25	Jan/88
Turkey	2.34	18.90	0.23	1.04	11.86	0.56	2.13	4.47	Jan/88
Venezuela	1.33	14.65	0.24	0.85	10.18	0.44	1.46	−0.23	Jan/93
Avg. DMs	**0.97**	**6.53**	**0.61**	**0.93**	**4.54**	**0.74**	**1.06**	**0.43**	**N/A**
Avg. EMs	**1.17**	**11.86**	**0.31**	**0.92**	**7.77**	**0.55**	**1.38**	**2.59**	**N/A**
Avg. all	**1.08**	**9.41**	**0.45**	**0.93**	**6.28**	**0.64**	**1.24**	**1.60**	**N/A**
World	**0.78**	**4.17**	**1.00**	**1.00**	**3.11**	**1.00**	**1.00**	**−2.14**	**Jan/88**

Table 11: Summary statistics – Estrada data set (monthly stock returns)

Full sample: simple regression analysis

$MR_i = \gamma_0 + \gamma_1 RV_i + u_i$

RV	γ_0	t-stat	γ_1	t-stat	R^2	Adj-R^2
Panel A: OLS estimation						
σ	0.06	0.28	0.11	4.87	0.33	0.32
β	0.11	0.48	1.04	4.49	0.30	0.28
Σ	−0.07	−0.27	0.18	5.01	0.34	0.33
β^D	−0.45	−1.83	1.24	6.58	0.47	0.46
Panel B: heteroskedasticity-consistent estimation						
σ	0.06	0.30	0.11	4.31	0.33	0.32
β	0.11	0.52	1.04	4.50	0.30	0.28
Σ	−0.07	−0.30	0.18	4.70	0.34	0.33
β^D	−0.45	−2.16	1.24	7.17	0.47	0.46

MR: mean return; RV: risk variable; σ: standard deviation; β: beta (with respect to the world market); Σ: semideviation; β^D: downside beta (with respect to the world market). Significance on Panel B based on White's heteroskedasticity-consistent covariance matrix. Critical value for a two-sided test at the 5% significance level: 2.01.

Table 12: Estrada simple regression analysis – Full sample

Full sample: multiple regression analysis

Panel A: $MR_i = \gamma_0 + \gamma_1 RV_{1i} + \gamma_2 RV_{2i} + v_i$

RV_1/RV_2	γ_0	t-stat	γ_1	t-stat	γ_2	t-stat	R^2
σ/Σ	−0.07	−0.29	−0.01	−0.05	0.19	0.78	0.34
β/β^D	−0.46	−2.15	−0.15	−0.37	1.35	3.83	0.48

Panel B: $MR_i = \gamma_0 + \gamma_1 RV_{1i} + \gamma_2 RV_{2i} + \gamma_3 RV_{3i} + \gamma_4 RV_{4i} + v_i$

$RV_1/RV_2/RV_3/RV_4$	γ_0	t-stat	γ_1	t-stat	γ_2	t-stat	γ_3	t-stat	γ_4	t-stat	R^2
$\sigma/\Sigma/\beta/\beta^D$	−0.48	−1.86	0.01	0.10	−0.55	−0.89	−0.16	−0.75	2.40	2.62	0.50

MR: mean return; RV: risk variable; σ: standard deviation; β: beta (with respect to the world market); Σ: semideviation; β^D: downside beta (with respect to the world market). Significance based on White's heteroskedasticity-consistent covariance matrix. Critical values for a two-sided test at the 5% significance level: 2.01 in both panels.

Table 13: Estrada multiple regression analysis – Full sample

DMs versus EMs: simple regression analysis

$MR_i = \gamma_0 + \gamma_1 RV_i + u_i$

RV	γ_0	t-stat	γ_1	t-stat	R^2	Adj-R^2
Panel A: DMs						
σ	0.63	1.95	0.05	1.12	0.06	0.01
β	0.76	1.99	0.22	0.55	0.01	−0.03
Σ	0.54	1.46	0.09	1.18	0.06	0.02
β^D	0.40	0.89	0.54	1.31	0.08	0.03
Panel B: EMs						
σ	−0.79	−1.81	0.16	4.79	0.48	0.46
β	0.11	0.34	1.14	3.75	0.36	0.33
Σ	−1.01	−2.20	0.28	5.00	0.50	0.48
β^D	−0.80	−2.11	1.42	5.57	0.55	0.54

MR: mean return; RV: risk variable; σ: standard deviation; β: beta (with respect to the world market); Σ: semideviation; β^D: downside beta (with respect to the world market). Critical values for a two-sided test at the 5% significance level: 2.08 and 2.06 in panels A and B, respectively.

Table 14: Estrada simple regression analysis – DMs vs EMs

DMs vs. EMs: multiple regression analysis

Panel A: $MR_i = \gamma_0 + \gamma_1 RV_{1i} + \gamma_2 RV_{2i} + v_i$

RV_1/RV_2	γ_0	t-stat	γ_1	t-stat	γ_2	t-stat	R^2
Panel A1: DMs							
σ/Σ	0.52	1.49	−0.03	−0.13	0.14	0.40	0.06
β/β^D	0.41	1.06	−0.56	−0.65	1.02	1.53	0.11
Panel A2: EMs							
σ/Σ	−1.00	−2.96	0.02	0.11	0.25	1.07	0.50
β/β^D	−1.01	−2.72	−0.65	−1.30	2.01	4.26	0.58

Panel B: $MR_i = \gamma_0 + \gamma_1 RV_{1i} + \gamma_2 RV_{2i} + \gamma_3 RV_{3i} + \gamma_4 RV_{4i} + v_i$

$RV_1/RV_2/RV_3/RV_4$	γ_0	t-stat	γ_1	t-stat	γ_2	t-stat	γ_3	t-stat	γ_4	t-stat	R^2
Panel B1: DMs											
$\sigma/\Sigma/\beta/\beta^D$	0.38	1.06	0.01	0.03	−0.94	−0.76	−0.17	−0.24	2.04	1.42	0.12
Panel B2: EMs											
$\sigma/\Sigma/\beta/\beta^D$	−0.99	−2.43	−0.02	−0.12	−0.72	−1.13	−0.00	−0.01	2.20	2.33	0.58

MR: mean return; RV: risk variable; σ: standard deviation; β: beta (with respect to the world market); Σ: semideviation; β^D: downside beta (with respect to the world market). Significance based on White's heteroskedasticity-consistent covariance matrix. Critical values for a two-sided test at the 5% significance level: 2.09, 2.06, 2.10, and 2.07 in panels A1, A2, B1, and B2, respectively.

Table 15: Estrada multiple regression analysis – DMs vs EMs

Laxmi C. Bhandari

Table 16

Estimated Coefficients (Percent per Month) for the Expected-Return Equation $[E(\tilde{r}_i) = \gamma_0 + \gamma_1 LTEQ_i + \gamma_2 BETA_i + \gamma_3 DER_i]$, Their Estimated Test-Period Betas, and the Summary Statistics for the Explanatory Variables: 1948–1979[a]

	All-Firms Sample				Manufacturing-Firms Sample			
	Constant	*LTEQ*	*BETA*	*DER*	Constant	*LTEQ*	*BETA*	*DER*
Panel A: Weighted (in Inverse Proportion to Estimated Standard Errors) Averages of the Subperiod Mean Estimates								
Including January	1.01	−0.10	0.30	0.09	1.29	−0.11	−0.05	0.17
t-Statistic		(−3.12)	(1.23)	(3.93)		(−3.14)	(−0.21)	(3.91)
Excluding January	0.56	−0.01	0.16	0.05	0.85	−0.02	−0.19	0.10
t-Statistic		(−0.45)	(0.60)	(2.37)		(−0.62)	(−0.76)	(2.24)
Panel B: Simple Averages of the Subperiod Mean Estimates								
Including January	1.01	−0.11	0.17	0.13	1.45	−0.12	−0.24	0.18
t-Statistic		(−3.00)	(0.60)	(3.84)		(−3.32)	(−0.87)	(3.47)
Excluding January	0.52	0.00	0.01	0.09	1.01	−0.02	−0.40	0.11
t-Statistic		(0.01)	(0.04)	(2.47)		(−0.57)	(−1.47)	(2.00)
Panel C: Estimated Test-Period Betas of the Estimated Coefficients (Including January)								
Estimated Test-Period Beta	0.07	−0.00	0.93	−0.00	0.18	−0.01	0.83	0.02
t-Statistic	(0.94)	(−0.23)	(−1.00)	(−0.08)	(2.10)	(−0.47)	(−2.39)	(0.82)

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