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A Comparative Analysis of Photon and Electron Wave Functions in Spherically Symmetric Potentials

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A Comparative Analysis of Photon and Electron Wave Functions in Spherically Symmetric Potentials

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Abstract

In this paper, we derived the wave function for light in a finite spherical well potential by drawing an analogy to an electron in an analogous potential energy well. In order to solve this problem, the theory behind vector calculus on a spherical basis was examined. The wave equation for light, derived from Maxwell's equations, and the Pauli equation for electrons were combined into a unified form that was solved separation of variables, infinite series solutions, and numerical methods. The potential well for light was established by considering an environment with a constant index of refraction inside a spherical boundary and a different, but still constant, index of refraction outside the boundary. It was determined that the radial part of the wave function oscillated more inside the boundary as radial quantum number increased. The distance between the origin and the first peak amplitude inside the boundary increased as the angular momentum quantum number increased. For light, the wave number was found to be complex, therefore the temporal part of the wave function was a dampened oscillator. The time constant of the temporal part of the wave function increased as angular momentum quantum number increased and decreased when the radial quantum number increased.

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My journey as a writer at the College of Wooster began in my First-Year Seminar with Dr. Daniel Bourne. Dr. Bourne taught me the importance of narrative and helped me come to the realization that I could continue to pursue my passion for writing as a STEM major. Oddly, I took the writing intensive course in both of majors with my IS advisors. The number theory course with Dr. Bowen gave me my first experience with LaTeX and in theoretical mathematics. I cannot stress enough how important that class was in affirming my decision to major in mathematics. Dr Leary's modern physics course showed me just how cool physics could be in the lecture portion and the lab portion of the class broadened my ability to think critically and write as a scientist. I had Dr. Lehman for Junior Independent Study lecture and laboratory and my self designed project was advised by Dr. Adam Fritsch and Dr. Karen Lewis. Over the course of Junior IS, Dr Lehman helped me realize that brevity is not necessarily the soul of wit. Dr. Lewis provided me with the knowledge base to approach a self designed project involving astronomy and Dr. Fritsch helped me to incorporate work that I had already done into the context of a new problem.

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imagine having class after class together, growing closer and closer. Imagine this person introducing you to an entirely new group of people and a cause worth fighting for. Imagine watching this person be a hero for so many people that were going through their darkest hour. Imagine getting to be the hero that is there to save them. Imagine leading multiple efforts together. Luckily for me, I don't imagine having someone this awesome in my life. I can imagine just about any theoretical mathematical or physical concept that I've came across, but the one thing that I cannot imagine is my life at Wooster without Maggie Lankford. Thank you Maggie. Thank you all.

Contents

List of Figures

Chapter 1

Introduction

Wave-particle duality is a fundamental principle of quantum mechanics. The idea that we can represent an object that is seemingly discrete and finite as a continuous wave with potentially infinite extent is a very unintuitive principle. Yet, over one hundred years later, the theory based on this paradoxical idea, quantum mechanics, is still integral to our understanding of physics today [4]. In this thesis, we will draw an analogy between photons, whose wave equations can be derived from classical physics, to electrons, whose wave nature follows directly from quantum mechanical principles.

Our investigation will begin where multivariable calculus courses often end: vector differential operations. In single variable calculus derivatives only take on one form, that is the ordinary derivative of a scalar function is a scalar function. But, when one considers functions with multiple dimensional domain and codomain, a myriad of possibilities arise. It is this mathematical richness that will allow us to tame our physics problem in an elegant manner.

As previously mentioned, this thesis will be comparing and contrasting

the wave nature of two fundamental particles: electrons and photons. These two particles are radically different. Electrons have mass, while photons are massless. Photons are spin one particles, but electrons are spin one-half particles. The differences between these two particles are numerous and certainly non-negligible.

In wave physics, we use wave functions to describe the behavior of objects and systems, such as fundamental particles [4]. The wave equation for a particle is the differential equations whose solutions are the wave functions. We will show that the wave equations for electrons and photons in a spherically symmetric potential can ultimately be massaged into a single unified form, and as such will have analogous solutions.

Obtaining solutions to the unified wave equation will require several advanced differential equations techniques. Since the unified wave equation is a partial differential equation, we will need to split the unified wave equation into a set of ordinary differential equations using separation of variables. In order to solve the set of ordinary differential equations that results from separation of variables, we will use infinite series and solutions to similar differential equations.

In order to obtain wave functions from a wave equation, we must solve a boundary value problem (BVP) [5]. That is, we must impose a set of conditions that functions in the solution set must meet in addition to being solutions to the wave equation. The implication of this statement is an important nuance in wave physics and BVPs in general; a function that solves the differential equation is not necessarily a solution to the BVP. Once we have derived the solutions to the BVP, we will analyze the behavior of a sample of

wave functions in two ways. We will graphically examine the behavior of the radial portion of various wave functions as angular momentum and solution order varies. The other method through which we will analyze the wave functions is through looking at amplitude decay as angular momentum and radial momentum varies.

Chapter 2

Vector Calculus

In this chapter we will review the four vector calculus differential operations as they are usually taught in multivariable calculus using rectangular coordinates. Then, we will describe these operations in a more compact form using the del operator. After this, we will generalize these operations to an arbitrary orthogonal basis before considering the specific case of spherical coordinates.

Throughout this thesis, we will use square brackets [ ] to denote function arguments and parentheses ( ) will be used exclusively for grouping terms for arithmetic. For all of the definitions in this chapter, let $f[x, y, z]$ be a scalar function and $\mathbf{v}[x, y, z] = \left\langle v_x[x, y, z], v_y[x, y, z], v_z[x, y, z] \right\rangle$ be a vector field in $\mathbb{R}^3$. Moving forward, we will not write the functional dependence of f or $\mathbf{v}$, but it is important to remember that f and the components of $\mathbf{v}$ are all functions of x, y, and z. We will also use the convention of using the symbol v to denote the magnitude of the vector $\mathbf{v}$, that is $v = \|\mathbf{v}\|$.

There are four differential operations from vector calculus that we will be

using. The gradient is an operation that produces a vector field from a scalar function [9].

Definition 1. *The* **gradient** *of a scalar function f, denoted by* **grad** $[f]$*, is given by,*

$$\mathbf{grad}\,[f] = \left\langle \partial_x f, \partial_y f, \partial_z f \right\rangle, \tag{2.1}$$

where $\partial_x f$ is the partial derivative of f with respect to x and so forth.

The divergence and the curl are both operations that act on a vector field. Computing the divergence of a vector field results in a scalar, while the curl of a vector field is another vector field [9].

Definition 2. *The* **divergence** *of a vector field* $\mathbf{v}$*, denoted by* **div** $[\mathbf{v}]$*, is given by,*

$$\mathbf{div}\,[\mathbf{v}] = \partial_x v_x + \partial_y v_y + \partial_z v_z. \tag{2.2}$$

Definition 3. *The* **curl** *of a vector field* $\mathbf{v}$*, denoted by* **curl** $[\mathbf{v}]$*, is given by,*

$$\mathbf{curl}\,[\mathbf{v}] = \left\langle \left(\partial_y v_z - \partial_z v_y\right), \left(\partial_z v_x - \partial_x v_z\right), \left(\partial_x v_y - \partial_y v_x\right) \right\rangle. \tag{2.3}$$

The last operation that we will discuss is the Laplacian, which acts on scalar function quantities to produce another scalar function [9].

Definition 4. *The* **Laplacian** *of a scalar function f, denoted by $\triangle f$, is given by,*

$$\triangle f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f, \tag{2.4}$$

where $\partial_x^2 f$ is the second partial derivative of f with respect to x twice and so forth.

Before we discuss the del operator, we will look at a cross product identity known as the *BAC CAB Rule*. This identity will prove useful in later calculations.

Theorem 1 (BAC CAB Rule)**.** *Let* $\mathbf{A} = \langle a_x, a_y, a_z \rangle$, $\mathbf{B} = \langle b_x, b_y, b_z \rangle$, *and* $\mathbf{C} = \langle c_x, c_y, c_z \rangle$ *such that* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^3$. *Then,*

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{2.5}$$

Proof. The proof of the BAC CAB rule is straightforward. Consider the product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$,

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \langle a_x, a_y, a_z \rangle \times \langle b_y c_z - b_z c_y, b_z c_x - b_x c_z, b_x c_y - b_y c_x \rangle && (2.6)\\
&= \langle a_y (b_x c_y - b_y c_x) - a_z (b_z c_x - b_x c_z), \\
&\quad a_z (b_y c_z - b_z c_y) - a_x (b_x c_y - b_y c_x), \\
&\quad a_x (b_z c_x - b_x c_z) - a_y (b_y c_z - b_z c_y) \rangle && (2.7)\\
&= \langle a_y b_x c_y - a_y b_y c_x - a_z b_z c_x + a_z b_x c_z, \\
&\quad a_z b_y c_z - a_z b_z c_y - a_x b_x c_y + a_x b_y c_x, \\
&\quad a_x b_z c_x - a_x b_x c_z - a_y b_y c_z + a_y b_z c_y \rangle && (2.8)\\
&= \langle a_y b_x c_y - a_y b_y c_x - a_z b_z c_x + a_z b_x c_z + a_x b_x c_x - a_x b_x c_x, \\
&\quad a_z b_y c_z - a_z b_z c_y - a_x b_x c_y + a_x b_y c_x + a_y b_y c_y - a_y b_y c_y, \\
&\quad a_x b_z c_x - a_x b_x c_z - a_y b_y c_z + a_y b_z c_y + a_z b_z c_z - a_z b_z c_z \rangle && (2.9)
\end{aligned}$$

$$
\begin{aligned}
&= \Big\langle b_x \left(a_x c_x + a_y c_y + a_z c_z\right) - c_x \left(a_x b_x + a_y b_y + a_z b_z\right), \\
&b_y \left(a_x c_x + a_y c_y + a_z c_z\right) - c_y \left(a_x b_x + a_y b_y + a_z b_z\right), \\
&b_z \left(a_x c_x + a_y c_y + a_z c_z\right) - c_z \left(a_x b_x + a_y b_y + a_z b_z\right) \Big\rangle && (2.10) \\
&= \Big\langle b_x \left(\mathbf{A} \cdot \mathbf{C}\right) - c_x \left(\mathbf{A} \cdot \mathbf{B}\right), b_y \left(\mathbf{A} \cdot \mathbf{C}\right) - c_y \left(\mathbf{A} \cdot \mathbf{B}\right), \\
&b_z \left(\mathbf{A} \cdot \mathbf{C}\right) - c_z \left(\mathbf{A} \cdot \mathbf{B}\right) \Big\rangle && (2.11) \\
&= \left\langle b_x, b_y, b_z \right\rangle \left(\mathbf{A} \cdot \mathbf{C}\right) - \left\langle c_x, c_y, c_z \right\rangle \left(\mathbf{A} \cdot \mathbf{B}\right) && (2.12) \\
&= \mathbf{B} \left(\mathbf{A} \cdot \mathbf{C}\right) - \mathbf{C} \left(\mathbf{A} \cdot \mathbf{B}\right). && (2.13)
\end{aligned}
$$

□

2.1 The Del Operator

We can restate the gradient, divergence, curl, and Laplacian in a more compact form by introducing the del operator ∇. The del operator is a vector operator whose components are derivatives in the corresponding coordinate. In $\mathbb{R}^3$ we have $\nabla = \left\langle \partial_x, \partial_y, \partial_z \right\rangle$ for Cartesian coordinates [5]. We define the action of ∇ on a function f to be to be $\nabla f = \left\langle \partial_x f, \partial_y f, \partial_z f \right\rangle$.

Theorem 2. *The four vector calculus differential operations, the gradient, divergence, curl, and Laplacian, are equivalent to applying the following vector operations using the del operator:*

$$\mathbf{grad}\left[f\right] = \nabla f \tag{2.14}$$

$$\mathbf{div}\left[\mathbf{v}\right] = \nabla \cdot \mathbf{v} \tag{2.15}$$

$$\mathbf{curl}\,[\mathbf{v}] = \nabla \times \mathbf{v} \tag{2.16}$$

$$\triangle f = \nabla \cdot \nabla f, \tag{2.17}$$

where we define the operation $\nabla \cdot \nabla f$ *to be* $\nabla^2 f$*.*

Proof. By definition $\nabla f = \left\langle \partial_x f, \partial_y f, \partial_z f \right\rangle = \mathbf{grad}\,[f]$, so we have verified Eqn. 2.14. Let us examine the divergence relationship. By computing we can see,

$$\nabla \cdot \mathbf{v} = \left\langle \partial_x, \partial_y, \partial_z \right\rangle \cdot \left\langle v_x, v_y, v_z \right\rangle \tag{2.18}$$

$$= \partial_x v_x + \partial_y v_y + \partial_z v_z \tag{2.19}$$

$$= \mathbf{div}\,[\mathbf{v}]\,. \tag{2.20}$$

Hence, Eqn. 2.15 has been verified. Next, we will prove the curl relationship:

$$\nabla \times \mathbf{v} = \left\langle \partial_x, \partial_y, \partial_z \right\rangle \times \left\langle v_x, v_y, v_z \right\rangle \tag{2.21}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ v_x & v_y & v_z \end{vmatrix} \tag{2.22}$$

$$= \left\langle \partial_y v_z - \partial_z v_y, \partial_z v_x - \partial_x v_z, \partial_x v_y - \partial_y v_x \right\rangle \tag{2.23}$$

$$= \mathbf{curl}\,[\mathbf{v}]\,. \tag{2.24}$$

Thus, we have verified Eqn. 2.16. Now we will verify the Laplacian

relationship,

$$\nabla^2 f = \nabla \cdot \nabla f \tag{2.25}$$

$$= \left\langle \partial_x, \partial_y, \partial_z \right\rangle \cdot \left\langle \partial_x f, \partial_y f, \partial_z f \right\rangle. \tag{2.26}$$

Recall that when we apply the divergence, we differentiate each component with respect to its corresponding variable, then sum the components. As such, the divergence of a gradient is a sum of second derivatives,

$$\nabla^2 f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f \tag{2.27}$$

$$= \triangle f. \tag{2.28}$$

With this, Eqn. 2.17 is verified. So we have shown that all four of our vector calculus differential operations can be expressed in terms of del. □

Now we will derive a product rule for the divergence of a vector field that has been multiplied by a function.

Theorem 3. *For a scalar function f and a vector field* $\mathbf{v}$,

$$\nabla \cdot (f\mathbf{v}) = f(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla f. \tag{2.29}$$

Proof. We can show that this theorem is true by applying the single variable product rule,

$$\nabla \cdot (f\mathbf{v}) = \nabla \cdot \left(f \left\langle v_x, v_y, v_z \right\rangle \right) \tag{2.30}$$

$$= \nabla \cdot \left\langle f v_x, f v_y, f v_z \right\rangle \tag{2.31}$$

$$= \partial_x (f v_x) + \partial_y \left(f v_y \right) + \partial_z (f v_z) \tag{2.32}$$

$$= v_x \partial_x f + f \partial_x v_x + v_y \partial_y f + f \partial_y v_y + v_z \partial_z f + f \partial_z v_z \tag{2.33}$$

$$= f \partial_x v_x + f \partial_y v_y + f \partial_z v_z + v_x \partial_x f + v_y \partial_y f + v_z \partial_z f \tag{2.34}$$

$$= f \left(\partial_x v_x + \partial_y v_y + \partial_z v_z \right) + \left\langle v_x, v_y, v_z \right\rangle \cdot \left\langle \partial_x f, \partial_y f, \partial_z f \right\rangle \tag{2.35}$$

By substituting in the definitions of $\mathbf{v}$, the divergence and the gradient into Eqn. 2.35, we have

$$\nabla \cdot (f\mathbf{v}) = f(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla f,$$

as desired. □

2.2 Vector Calculus on an Arbitrary Basis

Until now, we have only discussed the four vector differential operations in terms of rectangular coordinates in $\mathbb{R}^3$. We will solve problems that involve the Laplacian in spherical coordinates. In order to proceed, we will first derive relationships for three of the four vector differential operations for an arbitrary orthogonal basis. Then, we will consider the case of spherical coordinates specifically.

Let $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be an arbitrary orthogonal basis for $\mathbb{R}^3$. That is, any vector in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent and are all orthogonal to each other. Consider a vector function

$$\ell = \langle x[u, v, w], y[u, v, w], z[u, v, w] \rangle . \tag{2.36}$$

Note that $\{u, v, w\}$ are the variables that we associate with the directions of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, just as we traditionally associate $\{x, y, z\}$ with $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. We will calculate an infinitesimal change in ℓ by taking the *total differential* of ℓ.

Definition 5. *The* **total differential** *of a function ℓ in terms of the basis $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is given by,*

$$\mathrm{d}\ell = \partial_u \ell \mathrm{d}u + \partial_v \ell \mathrm{d}v + \partial_w \ell \mathrm{d}w. \tag{2.37}$$

Now, for each term in the right hand side of the total differential, we will divide and multiply by the magnitude of the respective partial derivative of ℓ,

$$\mathrm{d}\ell = \frac{\partial_u \ell}{||\partial_u \ell||}||\partial_u \ell||\mathrm{d}u + \frac{\partial_v \ell}{||\partial_v \ell||}||\partial_v \ell||\mathrm{d}v + \frac{\partial_w \ell}{||\partial_w \ell||}||\partial_w \ell||\mathrm{d}w. \tag{2.38}$$

The quantity $\partial_q \ell / ||\partial_q \ell||$ is equal to the unit vector $\hat{\mathbf{q}}$, where $q \in \{u, v, w\}$. We define the set of **coefficient functions** $\{f, g, h\}$ by

$$f = ||\partial_u \ell||, \tag{2.39}$$

$$g = ||\partial_v \ell||, \tag{2.40}$$

$$h = ||\partial_w \ell||. \tag{2.41}$$

Note that $f, g, h \neq 0$. If at least one of them were equal to zero, then the dimension of the span of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ would have to be less than three, which would contradict that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ spans $\mathbb{R}^3$. Substituting the expressions for f, g, and h into Eqn. 2.38 yields the expression for the *line element* in terms of our arbitrary basis $\mathcal{B}$.

Definition 6. *The* **line element** $\mathrm{d}\ell$ *of the arbitrary basis* $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *with coefficient functions* $\{f, g, h\}$ *is given by,*

$$\mathrm{d}\ell = f\mathrm{d}u\hat{\mathbf{u}} + g\mathrm{d}v\hat{\mathbf{v}} + h\mathrm{d}w\hat{\mathbf{w}}. \tag{2.42}$$

We will now use the line element in order to derive the gradient and the divergence in terms of an arbitrary basis.

Lemma 1. *In terms of an arbitrary basis* $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$*, the gradient of a scalar function* t *is given by,*

$$\nabla t = \frac{1}{f}\partial_u t\hat{\mathbf{u}} + \frac{1}{g}\partial_v t\hat{\mathbf{v}} + \frac{1}{h}\partial_w t\hat{\mathbf{w}}. \tag{2.43}$$

Proof. Consider the total differential of the function t,

$$\mathrm{d}t = \partial_u t\mathrm{d}u + \partial_v t\mathrm{d}v + \partial_w t\mathrm{d}w, \tag{2.44}$$

in terms of our basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. We can also express the total differential of a function in terms of its gradient and the line element,

$$\mathrm{d}t = \nabla t \cdot \mathrm{d}\ell = (\nabla t)_u f\mathrm{d}u + (\nabla t)_v g\mathrm{d}v + (\nabla t)_w h\mathrm{d}w, \tag{2.45}$$

where $(\nabla t)_u$ is the u component of ∇t and so forth [5]. By equating Eqn. 2.44 to

Eqn. 2.45, we get the following relationships,

$$(\nabla t)_u = \frac{1}{f}\partial_u t, \tag{2.46}$$

$$(\nabla t)_v = \frac{1}{g}\partial_v t, \tag{2.47}$$

$$(\nabla t)_w = \frac{1}{h}\partial_w t. \tag{2.48}$$

Hence, in terms of our arbitrary basis, the gradient of a function t can be expressed as

$$\nabla t = \frac{1}{f}\partial_u t\hat{\mathbf{u}} + \frac{1}{g}\partial_v t\hat{\mathbf{v}} + \frac{1}{h}\partial_w t\hat{\mathbf{w}}.$$

□

Next, we will examine the divergence of a vector vield using an arbitrary basis.

Lemma 2. *In terms of an arbitrary basis $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, the divergence of a vector field $\mathbf{A} = \langle A_u, A_v, A_w\rangle$ is given by,*

$$\nabla \cdot \mathbf{A} = \frac{1}{fgh}\left(\partial_u\left(ghA_u\right) + \partial_v\left(hfA_v\right) + \partial_w\left(fgA_w\right)\right). \tag{2.49}$$

Proof. This proof follows techniques presented in [5]. In order to construct the divergence in an arbitrary basis, we will seek to invoke the divergence theorem,

$$\iiint_V (\nabla \cdot \mathbf{A})\,\mathrm{d}\tau = \oiint_{\partial V} \mathbf{A} \cdot \mathrm{d}\mathbf{a} \tag{2.50}$$

where V is a volume bounded by a closed surface ∂V, $\mathrm{d}\tau$ is the volume element in our basis, and $\mathrm{d}\mathbf{a}$ is an area element in our basis [9], [5]. Consider

an infinitesimal "box" in our basis, that is an object whose sides are "straight lines" with respect to our coordinates as shown in Fig. 2.1. Note that our coordinates could be angles, so our "straight lines" could be curves. In particular, we will consider a box whose dimensions are du by dv by dw and is oriented in space such that the upper left corner on its front face is the point (u_0, v_0, w_0). We will construct the surface integral from the divergence theorem for our infinitesimal box. Then, we will equate that surface integral to the volume integral of the divergence theorem and deduce the divergence of $\mathbf{A}$.

First, we will compute the volume element in terms of our arbitrary basis. Note that by the definition of the line element, the distance between u_0 and $u_0 + \mathrm{d}u$ is $f\mathrm{d}u$, not simply du. Similarly, the distance between v_0 and $v_0 + \mathrm{d}v$ is $g\mathrm{d}v$ and the distance between w_0 and $w_0 + \mathrm{d}w$ is $h\mathrm{d}w$. Hence, the volume of our infinitesimal box is the product of these three distances,

$$\mathrm{d}\tau = fgh\mathrm{d}u\mathrm{d}v\mathrm{d}w. \tag{2.51}$$

Next we will look at the contribution of each face of the box to the surface integral in the divergence theorem. Consider the front face of the rectangle. The infinitesimal area vector $\mathrm{d}\mathbf{a}_{\mathbf{front}}$ will be in the $-\hat{\mathbf{u}}$ direction and its magnitude will simply be the area of the face $gh\mathrm{d}v\mathrm{d}w$,

$$\mathrm{d}\mathbf{a}_{\mathbf{front}} = -gh\mathrm{d}v\mathrm{d}w\hat{\mathbf{u}}\,|_{u=u_0}. \tag{2.52}$$

It is important to note that on the front face $u = u_0$, since the functions g and h

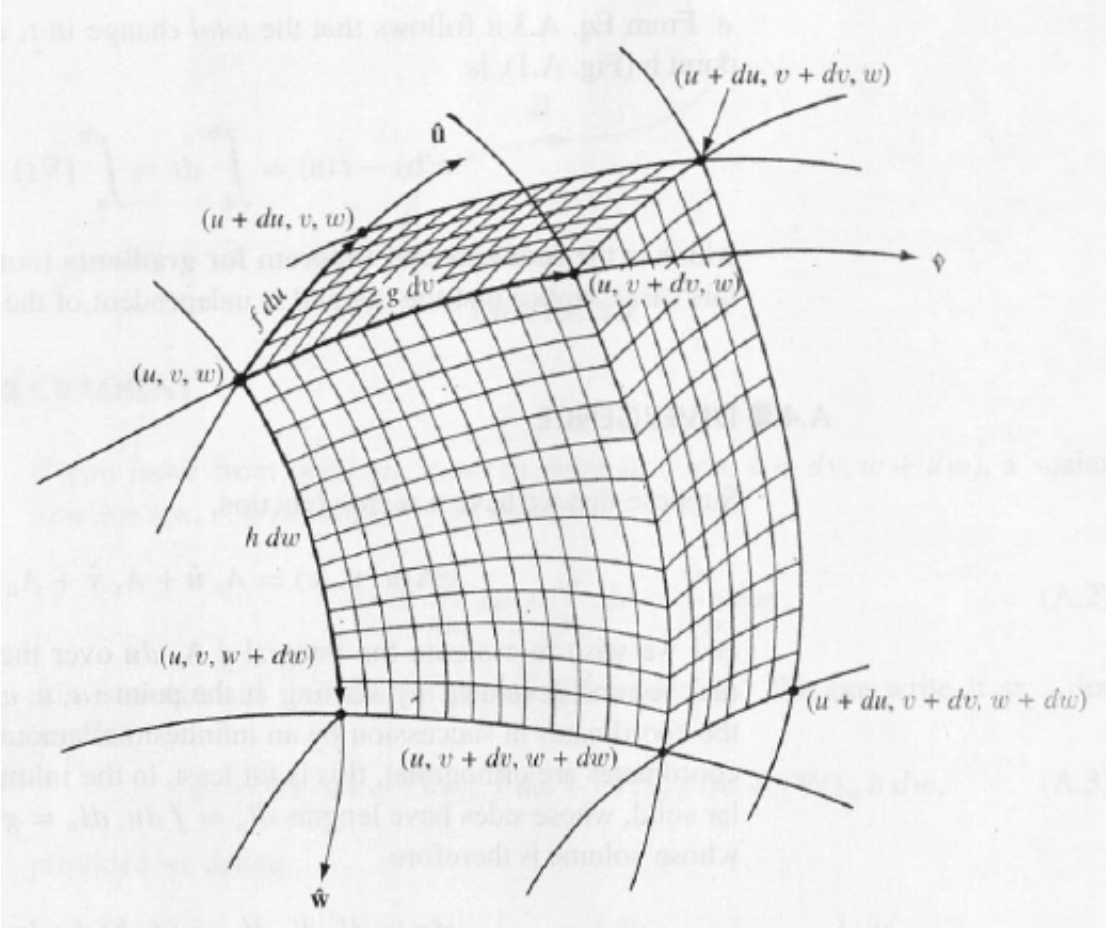

Figure 2.1: An infinitesimal "box" in a curvilinear coordinate system with basis $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Adapted from [5].

could depend on u. The area element for the back face of the box $\mathrm{d}\mathbf{a}_{\mathbf{back}}$ is similar to the area element for the front face of the box, except that it faces in the opposite direction and that $u = u_0 + \mathrm{d}u$ on the back face,

$$\mathrm{d}\mathbf{a}_{\mathbf{back}} = gh\mathrm{d}v\mathrm{d}w\hat{\mathbf{u}}\,|_{u=u_0+\mathrm{d}u}. \tag{2.53}$$

Constructing the integrands and differentials of the surface integrals for the

front and the back we have,

$$\begin{aligned}&\mathbf{A}\cdot\left(-gh\mathrm{d}v\mathrm{d}w\hat{\mathbf{u}}\right)|_{u=u_0}+\mathbf{A}\cdot\left(gh\mathrm{d}v\mathrm{d}w\hat{\mathbf{u}}\right)|_{u=u_0+\mathrm{d}u}\\&=-A_u gh\mathrm{d}v\mathrm{d}w|_{u=u_0}+A_u gh\mathrm{d}v\mathrm{d}w|_{u=u_0+\mathrm{d}u}.\end{aligned}\tag{2.54}$$

Now, we will use the following identity,

$$F\left[u+\mathrm{d}u\right]-F\left[u\right]=\partial_u F\mathrm{d}u\ ,\tag{2.55}$$

which is derived from the limit definition of the derivative [5]. By applying this identity to Eqn. 2.54 with $F=A_u gh\mathrm{d}v\mathrm{d}w$ we get,

$$A_u gh\mathrm{d}v\mathrm{d}w|_{u=u_0+\mathrm{d}u}-A_u gh\mathrm{d}v\mathrm{d}w|_{u=u_0}=\partial_u\left(A_u gh\mathrm{d}v\mathrm{d}w\right)\mathrm{d}u\tag{2.56}$$

$$=\partial_u\left(A_u gh\right)\mathrm{d}v\mathrm{d}w\mathrm{d}u.\tag{2.57}$$

Using Eqn. 2.51, we can substitute $\mathrm{d}\tau/\left(fgh\right)$ for $\mathrm{d}v\mathrm{d}w\mathrm{d}u$ into Eqn. 2.57,

$$\partial_u\left(A_u gh\right)\mathrm{d}v\mathrm{d}w\mathrm{d}u=\frac{1}{fgh}\partial_u\left(A_u gh\right)\mathrm{d}\tau.\tag{2.58}$$

Through analogous processes, it can be shown that the contribution of the left and right sides of the box to the surface integral is

$$\frac{1}{fgh}\partial_v\left(A_v hf\right)\mathrm{d}\tau$$

and that the contribution of the top and bottom side of the box to the surface

integral is

$$\frac{1}{fgh}\partial_w\left(A_w fg\right)\mathrm{d}\tau.$$

We have computed $\mathbf{A}\cdot\mathrm{d}\mathbf{a}$ for all six sides of the infinitesimal box. Hence, we can now construct the integral for the closed surface of the box by summing the integrals of our three integrands and differentials,

$$\oiint_{\partial V}\mathbf{A}\cdot\mathrm{d}\mathbf{a}=\iiint_V\frac{1}{fgh}\Big(\partial_u\left(A_u gh\right)+\partial_v\left(A_v hf\right)+\partial_w\left(A_w fg\right)\Big)\mathrm{d}\tau. \tag{2.59}$$

Thus, we have the integral of the dot product of a vector field $\mathbf{A}$ and an infinitesimal area vector $\mathrm{d}\mathbf{a}$ over a closed surface ∂V equal to the integral of a quantity over the volume V. By the Divergence Theorem, Eqn. 2.59 implies

$$\iiint_V\left(\nabla\cdot\mathbf{A}\right)\mathrm{d}\tau=\iiint_V\frac{1}{fgh}\Big(\partial_u\left(A_u gh\right)+\partial_v\left(A_v hf\right)+\partial_w\left(A_w fg\right)\Big)\mathrm{d}\tau. \tag{2.60}$$

The integrand of the right hand side must be the divergence of $\mathbf{A}$. Hence,

$$\nabla\cdot\mathbf{A}=\frac{1}{fgh}\left(\partial_u\left(ghA_u\right)+\partial_v\left(hfA_v\right)+\partial_w\left(fgA_w\right)\right), \tag{2.61}$$

as desired. □

Now that we have expressions for the gradient and the divergence in an arbitrary basis, we can derive an expression for the Laplacian in a arbitrary basis.

Theorem 4. *The Laplacian of a scalar function t, expressed in terms of an arbitrary*

basis $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$*, is given by,*

$$\nabla^2 t = \frac{1}{fgh}\left(\partial_u\left(\frac{gh}{f}\partial_u t\right) + \partial_v\left(\frac{hf}{g}\partial_v t\right) + \partial_w\left(\frac{fg}{h}\partial_w t\right)\right). \tag{2.62}$$

Proof. By Theorem 2, we have that $\nabla^2 t = \nabla \cdot \nabla t$. From Lemma 1, we have that in an arbitrary basis $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, the gradient of a scalar function is,

$$\nabla t = \frac{1}{f}\partial_u t\hat{\mathbf{u}} + \frac{1}{g}\partial_v t\hat{\mathbf{v}} + \frac{1}{h}\partial_w t\hat{\mathbf{w}}.$$

Now, we will take the divergence of this vector using Lemma 2,

$$\begin{aligned}
\nabla^2 t &= \nabla \cdot \nabla t && (2.63)\\
&= \frac{1}{fgh}\left(\partial_u\left(gh\frac{1}{f}\partial_u t\right) + \partial_v\left(hf\frac{1}{g}\partial_v t\right) + \partial_w\left(fg\frac{1}{h}\partial_w t\right)\right) && (2.64)\\
&= \frac{1}{fgh}\left(\partial_u\left(\frac{gh}{f}\partial_u t\right) + \partial_v\left(\frac{hf}{g}\partial_v t\right) + \partial_w\left(\frac{fg}{h}\partial_w t\right)\right). && (2.65)
\end{aligned}$$

Hence, the claim has been shown. □

2.3 Vector Calculus in Spherical Coordinates

Next, we will consider the specific case of when we choose spherical coordinates. The orthogonal basis $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ for spherical coordinates is $\{\hat{\mathbf{r}}, \hat{\theta}, \hat{\varphi}\}$. We can express the rectangular coordinates in terms of the spherical coordinates in the vector function

$$\ell = \langle r\sin[\theta]\cos[\varphi], r\sin[\theta]\sin[\varphi], r\cos[\theta]\rangle. \tag{2.66}$$

We will derive the coefficient functions for the line element,

$$\mathrm{d}\ell = f\mathrm{d}r\hat{\mathbf{r}} + g\mathrm{d}\theta\hat{\theta} + h\mathrm{d}\varphi\hat{\varphi}. \tag{2.67}$$

We will first compute the coefficient function f associated with the variable r by substituting into Eqn 2.39,

$$\begin{aligned}
f &= \|\partial_r \ell\| && (2.68)\\
&= \|\partial_r \langle r\sin[\theta]\cos[\varphi], r\sin[\theta]\sin[\varphi], r\cos[\theta]\rangle\| && (2.69)\\
&= \|\langle \sin[\theta]\cos[\varphi], \sin[\theta]\sin[\varphi], \cos[\theta]\rangle\| && (2.70)\\
&= \sqrt{\sin^2[\theta]\cos^2[\varphi] + \sin^2[\theta]\sin^2[\varphi] + \cos^2[\theta]} && (2.71)\\
&= \sqrt{\sin^2[\theta]\left(\cos^2[\varphi] + \sin^2[\varphi]\right) + \cos^2[\theta]} && (2.72)\\
&= \sqrt{\sin^2[\theta] + \cos^2[\theta]} && (2.73)\\
&= 1. && (2.74)
\end{aligned}$$

Similarly, we will compute the coefficient function g associated with the variable θ by substituting into Eqn 2.40,

$$\begin{aligned}
g &= \|\partial_\theta \ell\| && (2.75)\\
&= \|\partial_\theta \langle r\sin[\theta]\cos[\varphi], r\sin[\theta]\sin[\varphi], r\cos[\theta]\rangle\| && (2.76)\\
&= \|\langle r\cos[\theta]\cos[\varphi], r\cos[\theta]\sin[\varphi], -r\sin[\theta]\rangle\| && (2.77)
\end{aligned}$$

$$= \sqrt{r^2 \cos^2[\theta] \cos^2[\varphi] + r^2 \cos^2[\theta] \sin^2[\varphi] + r^2 \sin^2[\theta]} \tag{2.78}$$

$$= r\sqrt{\cos^2[\theta] \cos^2[\varphi] + \cos^2[\theta] \sin^2[\varphi] + \sin^2[\theta]} \tag{2.79}$$

$$= r\sqrt{\cos^2[\theta]\left(\cos^2[\varphi] + \sin^2[\varphi]\right) + \sin^2[\theta]} \tag{2.80}$$

$$= r\sqrt{\cos^2[\theta] + \sin^2[\theta]} \tag{2.81}$$

$$= r, \tag{2.82}$$

and the coefficient function h associated with the variable φ by substituting into Eqn 2.41,

$$h = \|\partial_\varphi \ell\| \tag{2.83}$$

$$= \|\partial_\varphi \langle r \sin[\theta] \cos[\varphi], r \sin[\theta] \sin[\varphi], r \cos[\theta] \rangle\| \tag{2.84}$$

$$= \|\langle -r \sin[\theta] \sin[\varphi], r \sin[\theta] \cos[\varphi], 0 \rangle\| \tag{2.85}$$

$$= \sqrt{r^2 \sin^2[\theta] \sin^2[\varphi] + r^2 \sin^2[\theta] \cos^2[\varphi]} \tag{2.86}$$

$$= r \sin[\theta] \sqrt{\sin^2[\varphi] + \cos^2[\varphi]} \tag{2.87}$$

$$= r \sin[\theta]. \tag{2.88}$$

Hence, the Laplacian in spherical coordinates can be computed by substituting into Eqn. 2.62,

$$\nabla^2 t = \frac{1}{1 \cdot r \cdot r \sin[\theta]} \left(\partial_r \left(\frac{r \cdot r \sin[\theta]}{1} \partial_r t \right) + \partial_\theta \left(\frac{r \sin[\theta] 1}{r} \partial_\theta t \right) + \partial_\varphi \left(\frac{1 \cdot r}{r \sin[\theta]} \partial_\varphi t \right) \right) \tag{2.89}$$

$$= \frac{1}{r^2 \sin[\theta]} \left(\partial_r \left(r^2 \sin[\theta] \partial_r t \right) + \partial_\theta \left(\sin[\theta] \partial_\theta t \right) + \partial_\varphi \left(\frac{1}{\sin[\theta]} \partial_\varphi t \right) \right) \tag{2.90}$$

$$= \frac{1}{r^2 \sin[\theta]} \left(\sin[\theta] \partial_r \left(r^2 \partial_r t \right) + \partial_\theta \left(\sin[\theta] \partial_\theta t \right) + \frac{1}{\sin[\theta]} \partial_\varphi \left(\partial_\varphi t \right) \right) \tag{2.91}$$

$$= \frac{1}{r^2} \partial_r \left(r^2 \partial_r t \right) + \frac{1}{r^2 \sin[\theta]} \partial_\theta \left(\sin[\theta] \partial_\theta t \right) + \frac{1}{r^2 \sin^2[\theta]} \partial_\varphi^2 t. \tag{2.92}$$

Thus, we have derived the Laplacian in spherical coordinates. This differential operator is some ways the mathematical crux of this thesis. We will be exploring the solutions to differential equations containing this operator. These solutions will then be applied in order to analyze physical systems that are modeled by differential equations containing the Laplacian in spherical coordinates.

Chapter 3

Preliminaries

In this chapter, we will derive a unified wave equation that describes the behavior of an electron placed in a given potential energy and a photon subject to an analogous potential due to the index of refraction of a medium. In order to do so, we will begin with the quantum mechanical description of the electron wave equation and Maxwell's equations for electromagnetic waves. Through substituting the definitions of relevant physical quantities, we will be able to write the electron wave equation and the photon wave equation in the same form.

3.1 Electron Wave Equation

In order to describe the wave functions of electrons in spherically symmetric potentials we will begin with the Pauli equation.

Definition 7. *The Pauli equation for a particle with mass m in a potential energy*

distribution $U[r]$ is given by,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U[r]\psi + H'\psi - E\psi = 0, \tag{3.1}$$

where $\hbar$ is the reduced Planck constant, ψ is the wave function of the particle, H' is a relativistic correction term, and E is the total energy of the particle [3].

We would like to isolate the Laplacian acting on ψ, so we multiply through by $-2m/\hbar^2$,

$$\nabla^2\psi - \frac{2m}{\hbar^2}U[r]\psi - \frac{2m}{\hbar^2}H'\psi + \frac{2m}{\hbar^2}E\psi = 0. \tag{3.2}$$

Now, we can multiply and divide the second term in our sum by the rest mass energy of the electron mc^2 [6],

$$\nabla^2\psi - \frac{2m^2c^2}{\hbar^2}\frac{U[r]}{mc^2}\psi - \frac{2m}{\hbar^2}H'\psi + \frac{2m}{\hbar^2}E\psi = 0, \tag{3.3}$$

where c is the speed of light in a vacuum. Next, we will express the constants in this expression in more succinct forms. The Compton wavelength of a particle λ_c is given by $h/(mc)$, where h is Planck's constant [6]. So we will define the *reduced Compton wavelength* $\lambda\!\!\!^{-}_c$ of a particle to be $\hbar/(mc)$,

$$\nabla^2\psi - \frac{2}{\lambda\!\!\!^{-2}_c}\frac{U[r]}{mc^2}\psi - \frac{2m}{\hbar^2}H'\psi + \frac{2m}{\hbar^2}E\psi = 0. \tag{3.4}$$

Since mc^2 has units of energy, the quantity $U[r]/(mc^2)$ is dimensionless. Hence we define the *dimensionless potential energy* for an electron $W_e[r]$ to be $U[r]/(mc^2)$,

$$\nabla^2\psi - \frac{2W_e[r]}{\lambda\!\!\!^{-2}_c}\psi - \frac{2m}{\hbar^2}H'\psi + \frac{2m}{\hbar^2}E\psi = 0. \tag{3.5}$$

We will define a new corrective term H'_e to be $-2mH'\hbar^{-2}$,

$$\nabla^2\psi - \frac{2W_e[r]}{\lambda\!\!\!^{-2}_c}\psi - H'_e\psi + \frac{2m}{\hbar^2}E\psi = 0. \tag{3.6}$$

The energy of a free particle is related to its momentum p by the expression $E = p^2/(2m)$. The momentum of a particle is given by $p = \hbar/\lambda\!\!\!^{-}_f$, where $\lambda\!\!\!^{-}_f$ is the reduced de Broglie wavelength of the particle [6]. Substituting these relationships into Eqn. 3.6 and simplifying yields,

$$\nabla^2\psi - \frac{2W_e[r]}{\lambda\!\!\!^{-2}_c}\psi - H'_e\psi + \frac{1}{\lambda\!\!\!^{-2}_f}\psi = 0. \tag{3.7}$$

Now we have a wave equation in terms of the Laplacian, dimensionless potential energy, wavelengths, and the corrective term all acting on the wave function. We will stop simplifying the electron case in isolation here and we will derive a similar form for the photon wave equation into in the next section.

3.2 Photon Wave Equation

Maxwell's equations are a set of partial differential equations that describe the electric field **E** and the magnetic field **B** present in a system given the charge density ρ and current density **J** present in that system [5].

Definition 8. *In free space,* **Maxwell's equations** *are:*

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \textit{Gauss's law} \tag{3.8}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \textit{Unnamed field equation} \tag{3.9}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \qquad \textit{Faraday's law} \tag{3.10}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial_t \mathbf{E} \qquad \textit{Ampére-Maxwell law,} \tag{3.11}$$

where ϵ_0 is the permittivity of free space and μ_0 is the permeability of free space.

Three of Maxwell's equations are named for the scientists that discovered them; Gauss, Faraday, Ampére, and Maxwell [5]. We will refer to Eqn. 3.9 as the **Equation of Magnetic Divergence**. We will be analyzing Maxwell's equations in a vacuum. In a vacuum, charge density is zero and the current density is the zero vector, so Maxwell's equations simplify,

$$\nabla \cdot \mathbf{E} = 0 \tag{3.12}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.13}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{3.14}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \partial_t \mathbf{E} \tag{3.15}$$

Note that Faraday's law and the Ampére-Maxwell law have both terms that depend on the electric field and terms that depend on the magnetic field. Maxwell's equations are system of four first order partial differential equations (two scalar equations and two vector equations, equivalent to a total of eight scalar equations) with both **E** and **B** as objective functions. This system can be

simplified into two vector second order partial differential equations (equivalent to six scalar equations), one with **E** as the objective function and one with **B** as the objective function [5]. Consider taking the curl of both sides of Eqn. 3.14,

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (-\partial_t \mathbf{B}). \tag{3.16}$$

Since the del operator only contains spatial derivatives, we can exchange the order of the time derivative and the curl in the right hand side of Eqn. 3.16. We can also apply the triple product "BAC CAB" rule to the left hand side of Eqn. 3.16,

$$\nabla(\nabla \cdot \mathbf{E}) - (\nabla \cdot \nabla)\mathbf{E} = -\partial_t(\nabla \times \mathbf{B}). \tag{3.17}$$

Now we can substitute Eqn. 3.12 and Eqn. 3.15 into Eqn. 3.17 and simplify,

$$\nabla(0) - (\nabla \cdot \nabla)\mathbf{E} = -\partial_t(\mu_0 \epsilon_0 \partial_t \mathbf{E}) \tag{3.18}$$

$$-\nabla^2 \mathbf{E} = -\partial_t(\mu_0 \epsilon_0 \partial_t \mathbf{E}) \tag{3.19}$$

$$-\nabla^2 \mathbf{E} = -\mu_0 \epsilon_0 \partial_t^2 \mathbf{E} \tag{3.20}$$

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \partial_t^2 \mathbf{E} = \mathbf{0}. \tag{3.21}$$

Hence, we have a second order partial differential equation depending only on **E**. Through an similar process, we can derive an identical second order partial differential equation for **B**,

$$\nabla^2 \mathbf{B} - \mu_0 \epsilon_0 \partial_t^2 \mathbf{B} = \mathbf{0}. \tag{3.22}$$

Now that we have simplified Maxwell's equations in a vacuum, we will

examine Maxwell's equations in media. In media, we can no longer assume that the permittivity ϵ and the permeability μ are the constants ϵ_0 and μ_0, respectively; we must consider ϵ and μ as functions of spatial variables. In order to account for this we need to define two new fields **D** and **H**,

$$\mathbf{D} = \epsilon \mathbf{E}, \tag{3.23}$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}. \tag{3.24}$$

Using these new fields, we can redefine Maxwell's equations.

Definition 9. *In media,* **Maxwell's equations** *are:*

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \textit{Gauss's law} \tag{3.25}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \textit{Unnamed field equation} \tag{3.26}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \qquad \textit{Faraday's law} \tag{3.27}$$

$$\nabla \times \mathbf{H} = \mathbf{J_f} + \partial_t \mathbf{D} \qquad \textit{Ampére-Maxwell law,} \tag{3.28}$$

where ρ_f is the **free charge density** *and $\mathbf{J_f}$ is the* **free current density** [5].

In a medium there are inherently charges and currents due to the nature of atoms. The charge and current densities that we care about for Maxwell's equations are the charge and current densities that are not negated by being paired with an equal and opposite charge or current density [5]. If all of the positive charges in an atom are paired with negative charges of equal magnitude, then the atom is electrically neutral and there is no free charge. Thus, we only want to consider unpaired charges for Maxwell's equations.

Similarly, it is the unpaired magnetic spins that lead to free current and therefore we only want to take into account currents resulting from unpaired magnetic spins. Hence, we refer to these unpaired current and charge densities as "free" current and charge densities in the definition of Maxwell's equations in media.

As previously stated, in general, ϵ and μ are arbitrary functions of spatial variables. For the systems that we will be considering, however, we can make simplifications. The media that we are concerned with will be non-ferrous, so we can set $\mu = \mu_0$. We will also only be considering cases when the permittivity is spherically symmetric. So if we choose to use spherical coordinates (r, θ, ϕ), then ϵ would depend only on r. Taking these assumptions into consideration, we will reduce Maxwell's equations in media from four first order partial differential equations that have **E** and **B** interdependent to a set of two second order partial differential equations, one equation for **E** and one equation for **B**. We will do so using similar methods to the reduction of Maxwell's equations in free space [5]. By taking the the curl of Eqn. 3.27 and applying derivative identities we get,

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\partial_t(\nabla \times \mathbf{B}). \tag{3.29}$$

In a vacuum, at this point we could substitute two other Maxwell equations for $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{B}$. In media, however, we do not have explicit equations for $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{B}$, we must solve for them. First we will solve for $\nabla \cdot \mathbf{E}$ beginning

with Eqn. 3.25 and substituting Eqn. 3.23,

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho_f \tag{3.30}$$

$$\epsilon(\nabla \cdot \mathbf{E}) + \mathbf{E} \cdot \nabla\epsilon = \rho_f \tag{3.31}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon}\left(\rho_f - \mathbf{E} \cdot \nabla\epsilon\right). \tag{3.32}$$

Note that here we have implicitly included our assumption that ϵ is dependent on a spatial variable by using Eqn 2.29, a vector calculus product rule, to expand Eqn. 3.30 into Eqn. 3.31. Similarly, we can also solve for $\nabla \times \mathbf{B}$ by substituting Eqn. 3.23 and Eqn. 3.24 into Eqn. 3.28,

$$\nabla \times \mathbf{H} = \mathbf{J_f} + \partial_t \mathbf{D}$$

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B}\right) = \mathbf{J_f} + \partial_t (\epsilon \mathbf{E}) \tag{3.33}$$

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J_f} + \epsilon \partial_t \mathbf{E} \tag{3.34}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J_f} + \mu_0 \epsilon \partial_t \mathbf{E}. \tag{3.35}$$

Once again, we have implicitly used our physical assumptions about μ and ϵ, setting $\mu = \mu_0$ and considering ϵ as a function of only space variable and hence constant with respect to a time derivative when going from Eqn. 3.33 to Eqn. 3.34. Now we can substitute Eqns. 3.32 and 3.35 into Eqn. 3.29 and

simplify,

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\partial_t(\nabla \times \mathbf{B})$$

$$\nabla\left(\frac{1}{\epsilon}\left(\rho_f - \mathbf{E}\cdot\nabla\epsilon\right)\right) - \nabla^2\mathbf{E} = -\partial_t\left(\mu_0\mathbf{J_f} + \mu_0\epsilon\partial_t\mathbf{E}\right) \tag{3.36}$$

$$\nabla\left(\frac{\rho_f}{\epsilon} - \frac{\mathbf{E}\cdot\nabla\epsilon}{\epsilon}\right) - \nabla^2\mathbf{E} = -\mu_0\partial_t\mathbf{J_f} - \mu_0\epsilon\partial_t^2\mathbf{E} \tag{3.37}$$

Note that since ϵ is a function of r only, the gradient of ϵ is simply $\partial_r\epsilon$. By the Chain Rule, $\partial_r \ln[\epsilon] = (\partial_r\epsilon)/\epsilon$. So in this case, $(\nabla\epsilon)/\epsilon = \nabla \ln[\epsilon]$. Substituting this relationship, we have:

$$\nabla^2\mathbf{E} - \nabla\left(\frac{\rho_f}{\epsilon} - \mathbf{E}\cdot\nabla\ln[\epsilon]\right) - \mu_0\epsilon\partial_t^2\mathbf{E} = \mu_0\partial_t\mathbf{J_f} \tag{3.38}$$

$$\nabla^2\mathbf{E} - \mu_0\epsilon\partial_t^2\mathbf{E} - \nabla\left(\frac{\rho_f}{\epsilon}\right) + \nabla\left(\mathbf{E}\cdot\nabla\ln[\epsilon]\right) = \mu_0\partial_t\mathbf{J_f} \tag{3.39}$$

$$\nabla^2\mathbf{E} - \mu_0\epsilon\partial_t^2\mathbf{E} + \nabla\left(\mathbf{E}\cdot\nabla\ln[\epsilon]\right) = \nabla\left(\frac{\rho_f}{\epsilon}\right) + \mu_0\partial_t\mathbf{J_f}. \tag{3.40}$$

Now if we assume that our system has zero free charge density and zero free current density, set $\rho_f = 0$ and $\mathbf{J_f} = \mathbf{0}$, the right hand side of Eqn. 3.40 becomes $\mathbf{0}$,

$$\nabla^2\mathbf{E} - \mu_0\epsilon\partial_t^2\mathbf{E} + \nabla\left(\mathbf{E}\cdot\nabla\ln[\epsilon]\right) = \mathbf{0}. \tag{3.41}$$

By comparing Eqn. 3.41, the second order partial differential equation for $\mathbf{E}$ in a media with no free current and no free charge, with Eqn. 3.21, the second order partial differential equation for $\mathbf{E}$ in a vacuum, we can see that the two equations are nearly identical. The only differences are that in media we have ϵ as a function (that we choose as an initial condition) and the addition of the term $\nabla\left(\mathbf{E}\cdot\nabla\ln[\epsilon]\right)$. We will use this additional term to define the **corrective**

term for photons $H'_\gamma \mathbf{E}$.

We choose H'_γ to represent the coefficient of the corrective term in order to draw a parallel between the mathematical structure of the photon differential wave equation to that of the electron differential wave equation. Now we perform further algebraic manipulation on the photon differential wave equation. We can express the temporal dependence of the electric field as $\mathbf{E}\left[r, \theta, \phi, t\right] = \mathbf{E}\left[r, \theta, \phi\right] e^{-i\omega t}$, where ω is the *temporal frequency of the wave* [5]. With this statement, we can evaluate the time derivative in Eqn. 3.41,

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon \omega^2 \mathbf{E} + H'_\gamma \mathbf{E} = \mathbf{0}. \tag{3.42}$$

The permittivity in media $\epsilon\,[r]$ is proportional to the square of the index of refraction of that media, $\epsilon\,[r] = \epsilon_0 n^2\,[r]$. By substituting this relationship along with the fact that the speed of light in a vacuum c is related to ϵ_0 and μ_0 by $\epsilon_0 \mu_0 = c^{-2}$ we have,

$$\nabla^2 \mathbf{E} - \frac{\omega^2 n^2\,[r]}{c^2} \mathbf{E} + H'_\gamma \mathbf{E} = \mathbf{0}. \tag{3.43}$$

Here we explicitly write the functional dependence of the index of refraction squared in preparation for a non-intuitive substitution. We will multiply the second term of our sum by $n^2\,[0]\,/n^2\,[0]$ (note that the index of refraction is never zero),

$$\nabla^2 \mathbf{E} - \frac{\omega^2 n^2\,[0]}{c^2} \frac{n^2\,[r]}{n^2\,[0]} \mathbf{E} + H'_\gamma \mathbf{E} = \mathbf{0}. \tag{3.44}$$

Now we define two quantities, the first is a function $W_\gamma\,[r] = n^2\,[r]\,/n^2\,[0]$. This term is analogous to a potential energy function for an electron, as it describes the index of refraction at a particular point r relative to a reference point 0.

Note that the collection of constants $(\omega n[0])/c$ has units of inverse wavelength. Hence, the wavelength of the photon in the medium is $c/(\omega n[0])$. We will use $\lambda_{\gamma 0}$ to denote this wavelength. Making these replacements, we have,

$$\nabla^2 \mathbf{E} - \frac{1}{\lambda_{\gamma 0}^2} W_\gamma[r]\,\mathbf{E} + H'_\gamma \mathbf{E} = \mathbf{0}. \tag{3.45}$$

Given the conditions of our problem, we can express $W_\gamma[r]$ in a manner that will allow us to draw a direct analogy to the form of the electron wave equation. We will do so in the next section.

3.3 Unifying Electron and Photon Wave Equations

Let us begin our comparative analysis of photon and electron behavior in spherically symmetric potentials by reminding ourselves of their respective wave equations. For photons we have Eqn. 3.45,

$$\nabla^2 \mathbf{E} - \frac{1}{\lambda_{\gamma 0}^2} W_\gamma[r]\,\mathbf{E} + H'_\gamma \mathbf{E} = \mathbf{0},$$

and for electrons we have Eqn. 3.7,

$$\nabla^2 \psi - \frac{2W_e[r]}{\lambda\!\!\!^{-2}_c}\psi - H'_e\psi + \frac{1}{\lambda\!\!\!^{-2}_f}\psi = 0,$$

where $\mathbf{E}$ is the electric field of the photon and ψ is the wave function of the electron.

Now we will relabel our objective functions. We will call the objective function for the electron wave equation ψ_e and the objective function for the

photon wave equation ψ_γ,

$$\nabla^2\psi_\gamma - \frac{1}{\lambda_{\gamma 0}^2} W_\gamma[r]\,\psi_\gamma + H'_\gamma\psi_\gamma = 0, \tag{3.46}$$

$$\nabla^2\psi_e - \frac{2W_e[r]}{\lambda\!\!\!^{-2}_c}\psi - H'_e\psi_e + \frac{1}{\lambda\!\!\!^{-2}_f}\psi_e = 0. \tag{3.47}$$

Note is that the electron wave equation has a scalar objective function, while the photon wave equation has a vector objective function. As can be derived from work presented in [7], the vector nature of **E** under a spherically symmetric potential does not depend on r or t. That is, we could write **E** as a scalar function dependent on only r and t multiplied by a vector function only dependent on θ and φ. Hence, it will be sufficient to treat the wave equation for **E** as a scalar equation for the sake of analyzing the radial and temporal behavior of **E**.

We will solve the wave equations with the corrective factors H'_e and H'_γ set equal to zero,

$$\nabla^2\psi_\gamma - \frac{1}{\lambda_{\gamma 0}^2} W_\gamma[r]\,\psi_\gamma = 0, \tag{3.48}$$

$$\nabla^2\psi_e - \frac{2W_e[r]}{\lambda\!\!\!^{-2}_c}\psi + \frac{1}{\lambda\!\!\!^{-2}_f}\psi_e = 0. \tag{3.49}$$

Our two wave equations are still not quite analogous, but by taking advantage of the spherical symmetry of the problem, we can make a substitution that will make the equations analogous. For electrons, the effective potential $W_e[r]$ is given by $U[r]/(mc^2)$. Since m and c are constants, the functional dependence of W_e on r will be directly proportional to the functional dependence of U on r.

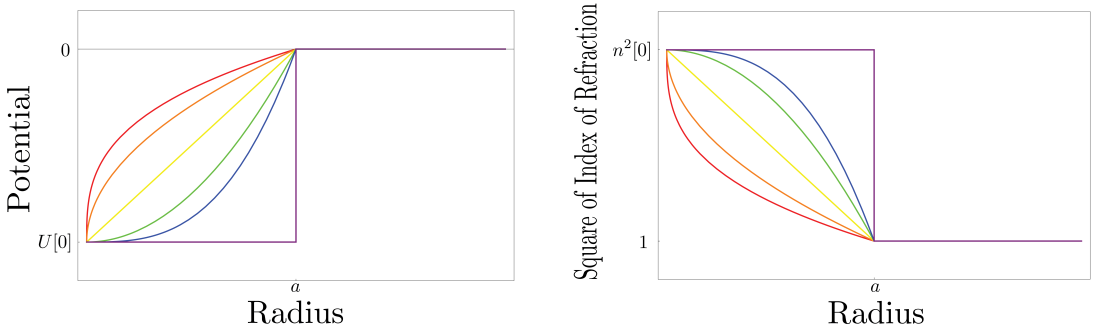

Figure 3.1: Graphs illustrating choices for $\chi[r]$ for electrons (left) and photons (right).

That is, if $U[r] = f[r]$, then $W_e[r] = \alpha f[r]$, where α is a proportionality constant. So we will work directly with the dependence of U on r and propagate our results to W_e. For photons we have a similar result, the effective potential $W_\gamma[r]$ is given by $n^2[r]/n^2[0]$. Since $n^2[0]$ is a constant, the functional dependence of W_γ on r is directly proportional to the functional dependence of $n^2[r]$ and thus we will work directly with the dependence of $n^2[r]$ on r.

For the photon, we will consider spheres in which the index of refraction squared is finite and greater than one at $r = 0$ and is equal to one at $r \geq a$, where a is the effective radius of the potential. For electrons, we will consider spherically symmetric potentials that are finite and less than zero at $r = 0$ and are equal to zero for $r \geq a$, where a is once again the radius of the sphere. For $0 < r < a$, the potential $U[r]$ and the index of refraction squared $n^2[r]$ could be any arbitrary function of r. In order to encapsulate this behavior we will introduce an abstract function $\chi[r]$ defined on $[0, a]$ with only two constraints, $\chi[0] = 0$ and $\chi[a] = 1$. The idea that $U[r]$ and $n^2[r]$ could be any arbitrary

function inside the boundary is illustrated by the two graphs in Fig. 3.1, with the different colored curves in the graphs corresponding to different choices for potential energy and index of refraction. Thus, each of the curves in the graphs of Fig. 3.1 also correspond to different choices for $\chi[r]$. Now, we will rewrite the potential and the index of refraction squared functions in terms of $\chi[r]$,

$$n^2[r] = n^2[0] + \chi[r]\left(n^2[a] - n^2[0]\right), \tag{3.50}$$

$$U[r] = U[0] + \chi[r]\left(U[a] - U[0]\right), \tag{3.51}$$

where $n^2[a] = 1$ and $U[a] = 0$. Note that the index of refraction and the electron potential still obey the previously discussed constraints. Next, by multiplying Eqn. 3.50 by $1/(mc^2)$ and Eqn. 3.51 by $1/(n^2[0])$, we obtain a new form for our effective potentials,

$$W_\gamma[r] = W_\gamma[0] + \chi[r]\left(W_\gamma[a] - W_\gamma[0]\right), \tag{3.52}$$

$$W_e[r] = W_e[0] + \chi[r]\left(W_e[a] - W_e[0]\right). \tag{3.53}$$

At this point, we can make a few simplifications. In Eqn. 3.52, we have $W_\gamma[0] = n^2[0]/(n^2[0]) = 1$ and we will define the quantity Δ_γ to be $W_\gamma[0] - W_\gamma[a]$. In Eqn. 3.53, we have $W_e[a] = U[a]/(mc^2) = 0$ and we will define the quantity Δ_e to be $-W_e[0]$. Substituting these relationships we have,

$$W_\gamma[r] = 1 - \Delta_\gamma\chi[r], \tag{3.54}$$

$$W_e[r] = -\Delta_e + \Delta_e\chi[r]. \tag{3.55}$$

We can now substitute the effective potentials back into the wave equations. Substituting Eqn. 3.54 into the photon wave equation Eqn. 3.48 and simplifying yields,

$$\nabla^2\psi_\gamma - \frac{1}{\lambda_{\gamma 0}^2}\psi_\gamma + \frac{\Delta_\gamma \chi[r]}{\lambda_{\gamma 0}^2}\psi_\gamma = 0. \tag{3.56}$$

Substituting Eqn. 3.55 into Eqn 3.49 and simplifying results in,

$$\nabla^2\psi_e + \frac{2\Delta_e \chi[r]}{\lambda\!\!\!^{-}_c^2}\psi_e - \frac{1}{\lambda\!\!\!^{-}_f^2}\psi_e - \frac{2\Delta_e}{\lambda\!\!\!^{-}_c^2}\psi_e = 0. \tag{3.57}$$

Recalling the definitions of $\lambda\!\!\!^{-}_f$, Δ_e, and $\lambda\!\!\!^{-}_c$ we can combine the last two terms of Eqn. 3.57,

$$\nabla^2\psi_e + \frac{2\Delta_e \chi[r]}{\lambda\!\!\!^{-}_c^2}\psi_e - \frac{2m}{\hbar}\left(\omega - \frac{mc^2 W_e[0]}{\hbar}\right)\psi_e = 0. \tag{3.58}$$

Note that the constant coefficients of the last term in our differential equation have units of inverse squared wavelength. We will define these constants to be $1/\left(\lambda_{e0}^2\right)$,

$$\nabla^2\psi_e + \frac{2\Delta_e \chi[r]}{\lambda\!\!\!^{-}_c^2}\psi_e - \frac{1}{\lambda_{e0}^2}\psi_e = 0. \tag{3.59}$$

Now we can finally express our differential equations in a single form,

$$\nabla^2\psi_i + \frac{\alpha_i \Delta_i \chi[r]}{\lambda_i^2}\psi_i - \frac{1}{\lambda_{i0}^2}\psi_i = 0, \tag{3.60}$$

where α_i is a constant and i is replaced by γ for the photon equation and by e in the electron equation ($\alpha_\gamma = 1$ and $\alpha_e = 2$). The general form of a wave equation is $\nabla^2\psi - k^2\psi = 0$; as such we will define,

$$k_i^2 = -\frac{\alpha_i \Delta_i \chi[r]}{\lambda_i^2}\psi_i + \frac{1}{\lambda_{i0}^2}, \tag{3.61}$$

and substitute this definition to arrive at the final form for the differential wave equation for both electrons and photons,

$$\nabla^2\psi_i - k_i^2\psi_i = 0. \tag{3.62}$$

Note that k_i is only a function of r. This will become important when we separate variables.

In this chapter, we derived the wave equation for photons using Maxwell's equations. We also manipulated both the photon wave equation and the Schroedinger equation, which describes the wave nature of electrons, into an identical form. Now, we are able to solve a single differential equation and gain intuition about the photon problem by drawing analogies to the electron case.

Chapter 4

Differential Equations Techniques

In this chapter, we examine three techniques that are used in solving differential equations. First, we will explore *separation of variables*, a technique that is used in order to turn a single partial differential equation into a set of ordinary differential equations. For the other two techniques, we will use differential equations that will appear during the process of solving Eqn. 3.62 as examples. We will use *infinite series solutions* in order to solve *Legendre's differential equation*. Then, we will use the solutions of the *Bessel differential equation* to derive the *associated solutions* of the *Spherical Bessel differential equation*

4.1 Separation of Variables

Definition 10. Linear homogeneous partial differential equations *(LHPDEs) are differential equations with objective function of n variables* $y[x_1, x_2, \ldots, x_n]$ *that have only a linear dependence derivatives of y and have no terms that are purely*

functions of $\{x_1, x_2, \ldots, x_n\}$.

In particular, we will be considering LHPDEs in which there are no terms that involve mixed partial derivatives of the objective function. That is, we will be looking at differential equations of the form

$$0 = \sum_{i=1}^{n} \sum_{k=0}^{m} q_{k_i} [x_1, x_2, \ldots, x_n] \, \partial_{x_i}^k y, \tag{4.1}$$

where m is the highest order derivative that our differential equation has and q_{k_i} is a function that is the coefficient for the k^{th} derivative of y with respect to x_i k times. For example, in the LHPDE

$$0 = \left(x_1^2 + x_2\right) \partial_{x_1}^2 y + x_1 \partial_{x_2} y, \tag{4.2}$$

y is a function of two variables, so $n = 2$. The highest order derivative in Eqn. 4.2 is 2, thus $m = 2$. The coefficient functions are $q_{2_1} = x_1^2 + x_2$, $q_{1_1} = 0$, $q_{2_2} = 0$, and $q_{1_2} = x_1$.

Theorem 5. *Let* $\mathbb{F}$ *be a field and let* $D[\mathbb{F}^n \to \mathbb{F}]$ *be the set of all infinitely differentiable functions from* $\mathbb{F}^n$ *to* $\mathbb{F}$. $D[\mathbb{F}^n \to \mathbb{F}]$ *is a vector space [8]. Vector addition is defined as function addition and scalar multiplication defined to be the multiplication operation of the field* $\mathbb{F}$.

In our case, we will have $\mathbb{R}$ as the field. The solutions to an LHPDE ζ must be differentiable functions. Let $S[\zeta]$ be the set of all solutions to ζ. Clearly, $S[\zeta]$ is a subset of the set $D[\mathbb{R}^n \to \mathbb{R}]$. In fact, $S[\zeta]$ forms a subspace of $D[\mathbb{R}^n \to \mathbb{R}]$. The method for solving LHPDEs that we are about to explore is also a method for forming a basis of the vector space $S[\zeta]$.

Separation of variables is a technique that can be used to solve some LHPDEs. When performing separation of variables, we assume a solution of the form

$$y[x_1, x_2, \ldots, x_n] = \mathcal{X}_1[x_1]\,\mathcal{X}_2[x_2]\ldots\mathcal{X}_n[x_n]. \tag{4.3}$$

Solutions to an arbitrary LHPDE ζ of this form are called the *product solutions* of ζ. Note that the product solutions of ζ are not the only solutions to ζ, but they do form a basis for $S[\zeta]$. That is, we can construct any solution of ζ by using a linear combination of the product solutions of ζ. After assuming a product solution and substituting this assumption, we then perform algebraic manipulations on ζ seeking to put ζ in the form

$$\sum_{k=0}^{m} q_{k_1}[x_1]\,\partial_{x_1}^k \mathcal{X}_1[x_1] + \sum_{i=2}^{n}\sum_{k=0}^{m} q_{k_i}[x_2, \ldots, x_n]\,\partial_{x_i}^k \mathcal{X}_2[x_2]\ldots\mathcal{X}_n[x_n] = 0. \tag{4.4}$$

If we cannot algebraically manipulate the LHPDE ζ into the form of Eqn 4.4, then we say that ζ is *non-separable*. Let λ and ρ be defined by

$$\lambda = \sum_{k=0}^{m} q_{k_1}[x_1]\,\partial_{x_1}^k \mathcal{X}_1[x_1] \tag{4.5}$$

$$\rho = \sum_{i=2}^{n}\sum_{k=0}^{m} q_{k_i}[x_2, \ldots, x_n]\,\partial_{x_i}^k \mathcal{X}_2[x_2]\ldots\mathcal{X}_n[x_n]. \tag{4.6}$$

Observe that the λ is a function of only x_1 and that ρ is not a function of x_1, but ρ is a function of $\{x_2, \ldots, x_n\}$. Consider the partial derivative of ρ with respect to x_1. Since ρ does not depend on x_1, $\partial_{x_1}\rho = 0$. Thus, ρ must be a constant. Similarly, by considering the partial derivative of λ with respect to x_2, it is clear that λ must be a constant. In fact, the sum of these constants must equal

zero, that is $\rho = -\lambda$. Substituting this result into Eqn 4.6, we have

$$-\lambda = \sum_{i=2}^{n} \sum_{k=0}^{m} q_{k_i} [x_2, \ldots, x_n] \, \partial_{x_i}^k \mathcal{X}_2 [x_2] \ldots \mathcal{X}_n [x_n] . \tag{4.7}$$

We call λ the **separation constant** of this LHPDE. Notice that we have separated the LHPDE ζ into two equations. We now have Eqn 4.5, an ordinary differential equation, and Eqn 4.7, an LHPDE with $n - 1$ variables. Next, we would seek to separate Eqn 4.7 into an ordinary differential equation and a new LHPDE with $n - 2$ variables. For an LHPDE with n variables, we seek to perform separation of variables $n - 1$ times. The net result of separation of variables is turning an LHPDE with n variables into n ordinary differential equations.

4.2 Series Solutions and Legendre's Equation

Legendre's differential equation is

$$\left(1 - x^2\right) \partial_x^2 y - 2x \partial_x y + \ell \left(\ell + 1\right) y = 0, \tag{4.8}$$

where $\ell \in \mathbb{Z}^+$ and $\mathbb{Z}^+$ is the set of all nonnegative integers. In order to solve Legendre's differential equation, we will need to take arbitrarily large derivatives of a product of functions. Therefore, it will be useful to recall *Leibniz' rule* for differentiation.

Definition 11 (Leibniz' rule)**.** *The n-th derivative with respect to x of a product of*

functions $f[x]$ *and* $g[x]$ *is*

$$\partial_x^n(fg) = \sum_{i=0}^{n} \binom{n}{i} \partial_x^i f \partial_x^{n-i} g. \tag{4.9}$$

Now, we are ready to seek solutions to Legendre's differential equation.

Theorem 6. *The* **Legendre polynomials** *are solutions to Legendre's differential equation and are given by* **Rodrigues's formula**,

$$P_\ell = \frac{1}{2^\ell \ell!} \partial_x^\ell \left(x^2 - 1\right)^\ell. \tag{4.10}$$

Proof. The techniques used in this proof are presented in [1]. We can solve this differential equation by looking for a power series solution, so let y, $\partial_x y$ and $\partial_x^2 y$ be the following,

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots \tag{4.11}$$

$$\partial_x y = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \ldots \tag{4.12}$$

$$\partial_x^2 y = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots + n\,(n+1)\,a_n x^{n-2} = \ldots, \tag{4.13}$$

where a_i is a constant for all $i \in \mathbb{N}$. Substituting Eqn. 4.11, Eqn. 4.12, and Eqn. 4.13 into Eqn. 4.8 and combining like terms we get

$$\begin{aligned} 0 = {} & \left(2a_2 + \ell\,(\ell+1)\,a_0\right) x^0 + \\ & \left(6a_3 + \left(\ell^2 + \ell - 2\right) a_1\right) x^1 + \\ & \left(12a_4 + \left(\ell^2 + \ell - 6\right) a_2\right) x^2 + \cdots + \\ & \left((n+2)\,(n+1)\,a_{n+2} + (\ell - n)\,(\ell + n + 1)\,a_n\right) x^n + \ldots. \end{aligned} \tag{4.14}$$

By the Fundamental Theorem of Algebra, if two polynomials are equal, then their coefficients must be equal identically. Hence, the x^n term of Eqn. 4.14 implies that,

$$a_{n+2} = -\frac{(\ell - n)(\ell + n + 1)}{(n+2)(n+1)} a_n. \tag{4.15}$$

So, we can recursively define all of the even coefficients in terms of a_0 and the odd coefficients in terms of a_1 in Eqn. 4.11,

$$\begin{aligned} y = a_0 &\left(1 - \frac{(\ell - 0)(\ell + 1)}{2 \cdot 1} x^2 + \left(-\frac{(\ell - 0)(\ell + 1)}{2 \cdot 1}\right) \cdot \left(-\frac{(\ell - 2)(\ell + 3)}{4 \cdot 3}\right) x^4 + \dots\right) \\ + a_1 &\left(x - \frac{(\ell - 1)(\ell + 2)}{3 \cdot 2} x^3 + \left(-\frac{(\ell - 1)(\ell + 2)}{3 \cdot 2}\right) \cdot \left(-\frac{(\ell - 3)(\ell + 4)}{5 \cdot 4}\right) x^5 + \dots\right) \end{aligned} \tag{4.16}$$

$$\begin{aligned} = a_0 &\left(1 - \frac{\ell(\ell + 1)}{2!} x^2 + \frac{\ell(\ell + 1)(\ell - 2)(\ell + 3)}{4!} x^4 - \dots\right) \\ + a_1 &\left(x - \frac{(\ell - 1)(\ell + 2)}{3!} x^3 + \frac{(\ell - 1)(\ell + 2)(\ell - 3)(\ell + 4)}{5!} x^5 - \cdots\right). \end{aligned} \tag{4.17}$$

For $\ell = 0$, the terms in the series expansion of y with a_1 as a coefficient are

$$x - \frac{(-1) \cdot (2)}{3!} x^3 + \frac{(-1) \cdot (2) \cdot (-3) \cdot (4)}{5!} x^5 - \dots = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \tag{4.18}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}. \tag{4.19}$$

This series diverges, therefore $a_1 = 0$. The terms with a_0 as a coefficient are zero, except the x^0 term, so $P_0[x] = 1$. Similarly, for $\ell = 1$, the terms of the

series expansion of y with a_0 as a coefficient are

$$1 - \frac{(1)\cdot(2)}{2!}x^2 + \frac{(1)\cdot(2)\cdot(-1)\cdot(4)}{4!}x^4 - \ldots = 1 - \frac{x^1}{1} - \frac{x^4}{3} - \ldots \tag{4.20}$$

$$= \sum_{n=0}^{\infty} - \frac{1}{2n-1}x^{2n}. \tag{4.21}$$

This series diverges, so $a_0 = 0$. The terms with a_1 as a coefficient are zero, except the x^1 term, so $P_1[x] = x$. In general, for even values of ℓ the odd terms in the series diverge and the terms of the even sequence that are of order greater than ℓ are zero and for odd values of ℓ the even terms in the series diverge and the terms of the odd sequence that are of order greater than ℓ are zero. Note that for negative values of ℓ, we obtain redundant solutions [1]. Thus, it makes sense that we only consider ℓ to be a nonnegative integer. So we have a set of polynomials, the Legendre polynomials, that solve the Legendre differential equation. Below we list the first four Legendre polynomials:

$$P_0[x] = 1 \tag{4.22}$$

$$P_1[x] = x \tag{4.23}$$

$$P_2[x] = \frac{3}{2}x^2 - \frac{1}{2} \tag{4.24}$$

$$P_3[x] = \frac{5}{2}x^3 - \frac{3}{2}x. \tag{4.25}$$

Now we will show that Rodrigues' formula serves as a generating function for Legendre Polynomials. First we will consider the function

$$v[x] = \left(x^2 - 1\right)^{\ell}, \tag{4.26}$$

We will show that the quantity $\partial_x^\ell v$ is a solution to the Legendre differential equation, up to a constant. Consider $(x^2-1)\,\partial_x v$,

$$\begin{aligned}\left(x^2-1\right)\partial_x v &= \left(x^2-1\right)\ell\left(x^2-1\right)^{\ell-1}\cdot 2x \\ &= 2\ell x v. \end{aligned} \tag{4.27}$$

Next, we can differentiate Eqn. 4.27 $\ell+1$ times using Leibniz' rule,

$$\partial_x^{\ell+1}\left(\left(x^2-1\right)\partial_x v\right) = \partial_x^{\ell+1}\left(2\ell x v\right) \tag{4.28}$$

$$\sum_{i=0}^{\ell+1}\binom{\ell+1}{i}\partial_x^i\left(x^2-1\right)\partial_x^{\ell+1-i}\partial_x v = 2\ell\sum_{i=0}^{\ell+1}\binom{\ell+1}{i}\partial_x^i x\partial_x^{\ell+1-i}v \tag{4.29}$$

Note that for for $i\geq 3$ the terms of the sum on the left hand side of Eqn. 4.29 are zero since $\partial_x^i\left(x^2-1\right)=0$ for $i\geq 3$. Similarly, the terms in the sum right hand side of Eqn. 4.29 are zero for $i\geq 2$ as $\partial_x^i x = 0$ for $i\geq 2$. Thus, expanding the sums yields

$$\left(x^2-1\right)\partial_x^{\ell+2}v + 2x\left(\ell+1\right)\partial_x^{\ell+1}v + \ell\left(\ell+1\right)\partial_x^{\ell+1}v = 2\ell x\partial_x^{\ell+1}v + 2\ell\left(\ell+1\right)\partial_x^\ell v. \tag{4.30}$$

We can further simplify this equation to obtain the Legendre differential equation with $\partial_x^\ell v$ as the objective function,

$$0 = -\left(x^2-1\right)\partial_x^{\ell+2}v + \left(2\ell x - 2x\left(\ell+1\right)\right)\partial_x^{\ell+1}v + \left(2\ell\left(\ell+1\right) - \ell\left(\ell+1\right)\right)\partial_x^\ell v \tag{4.31}$$

$$= \left(1-x^2\right)\partial_x^{\ell+2}v - 2x\partial_x^{\ell+1}v + \ell\left(\ell+1\right)\partial_x^\ell v \tag{4.32}$$

$$= \left(1-x^2\right)\partial_x^2\left(\partial_x^\ell v\right) - 2x\partial_x\left(\partial_x^\ell v\right) + \ell\left(\ell+1\right)\partial_x^\ell v. \tag{4.33}$$

Thus, we have shown that the functional dependence of Rodrigues' formula on x, $\partial_x^\ell v$, is a solution to the Legendre differential equation. As a normalization condition, we require that $P_\ell[1] = 1$ for all $\ell \in \mathbb{Z}^+$. Multiplying $\partial_x^\ell (x^2-1)^\ell$ by the constant $\left(2^\ell \ell!\right)^{-1}$ satisfies this normalization as shown in [1]. □

4.3 Associated Solutions and Bessel's Equation

The *Bessel differential equation* is

$$x^2 \partial_x^2 y + x \partial_x y + \left(x^2 - p^2\right) y = 0, \tag{4.34}$$

where p is a constant. It is important to note that p is not necessarily an integer. Before we discuss solutions to this differential equation, recall the *Gamma Function* Γ,

Definition 12. *The* **Gamma Function** $\Gamma[x]$ *is given by [2],*

$$\Gamma[x] = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t. \tag{4.35}$$

The solutions to the Bessel differential equation are the *Bessel functions* [1].

Definition 13. *The* **Bessel functions of the first kind** J_p *and the* **Bessel functions of the second kind** N_p *are given by*

$$J_p[x] = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma[n+1]\,\Gamma[n+1+p]} \left(\frac{x}{2}\right)^{2n+p}, \tag{4.36}$$

$$N_p[x] = \frac{1}{\sin[p\pi]} \left(\cos[p\pi]\, J_p[x] - J_{-p}[x]\right). \tag{4.37}$$

These solutions can be derived using infinite series as shown in [1]. Since the Bessel differential equation is a second order ordinary differential equation, we expect two linearly independent solutions and, in fact, J_p and N_p are linearly independent functions [1].

The *Spherical Bessel differential equation* is

$$x^2\partial_x^2 y + 2x\partial_x y + \left(x^2 - \ell\left(\ell + 1\right)\right) y = 0. \tag{4.38}$$

In order to derive the relationship between the solutions to the Spherical Bessel differential equation and the ordinary Bessel functions, we will begin with the Spherical Bessel differential equation. Consider a solution to the Spherical Bessel differential equation of the form

$$y\left[x\right] = f\left[x\right] \cdot x^{-1/2}, \tag{4.39}$$

where $f\left[x\right]$ is an arbitrary function of x. We seek to place constraints on f. Now, we will compute the first and second derivative of y with respect to x,

$$\partial_x y = x^{-1/2}\partial_x f - \frac{1}{2}x^{-3/2} f, \tag{4.40}$$

$$\partial_x^2 y = x^{-1/2}\partial_x^2 f - \frac{1}{2}x^{-3/2}\partial_x f - \frac{1}{2}x^{-3/2}\partial_x f + \frac{3}{4}x^{-5/2} f \tag{4.41}$$

$$= x^{-1/2}\partial_x^2 f - x^{-3/2}\partial_x f + \frac{3}{4}x^{-5/2} f. \tag{4.42}$$

Now we will substitute Eqn. 4.39, Eqn. 4.40, and Eqn. 4.42 into Eqn. 4.38 and

simplify,

$$
\begin{aligned}
0 &= x^2\left(x^{-1/2}\partial_x^2 f - x^{-3/2}\partial_x f + \frac{3}{4}x^{-5/2} f\right) + 2x\left(x^{-1/2}\partial_x f - \frac{1}{2}x^{-3/2} f\right) \\
&\quad + \left(x^2 - \ell\left(\ell+1\right)\right)\left(f \cdot x^{-1/2}\right) && (4.43) \\
&= x^2\left(\partial_x^2 f - x^{-1}\partial_x f + \frac{3}{4}x^{-2} f\right) + 2x\left(\partial_x f - \frac{1}{2}x^{-1} f\right) \\
&\quad + \left(x^2 - \ell\left(\ell+1\right)\right) f && (4.44) \\
&= x^2\partial_x^2 f - x\partial_x f + \frac{3}{4} f + 2x\partial_x f - f + \left(x^2 - \ell^2 - \ell\right) f. && (4.45) \\
&= x^2\partial_x^2 f + \left(2x - x\right)\partial_x f + \left(\frac{3}{4} - 1 + x^2 - \ell^2 - \ell\right) f. && (4.46) \\
&= x^2\partial_x^2 f + x\partial_x f + \left(x^2 - \left(\ell^2 + \ell + \frac{1}{4}\right)\right) f. && (4.47) \\
&= x^2\partial_x^2 f + x\partial_x f + \left(x^2 - \left(\ell + \frac{1}{2}\right)^2\right) f. && (4.48)
\end{aligned}
$$

Observe that Eqn 4.48 is Eqn 4.38, the Bessel differential equation, with $p = \ell + 1/2$. Therefore, the allowed choices for f are the linear combinations of the Bessel functions. Hence the allowed forms of y are Bessel functions multiplied by $x^{-1/2}$. As with the Bessel functions, the Spherical Bessel functions have two kinds, denoted by j_ℓ for the first kind and n_ℓ for the second kind. As a convention, we choose the following:

Definition 14. *The* **Spherical Bessel functions** *of the first kind j_ℓ and second kind n_ℓ are given by*

$$
\begin{aligned}
j_\ell\left[x\right] &= \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}\left[x\right], && (4.49) \\
n_\ell\left[x\right] &= \sqrt{\frac{\pi}{2x}} N_{\ell+1/2}\left[x\right]. && (4.50)
\end{aligned}
$$

Also note that just as with the Bessel functions, j_ℓ and n_ℓ are linearly independent.

Another equivalent set of linearly independent solutions to the Spherical Bessel differential equation are the *Spherical Hankel functions*, which are given by,

$$h_\ell^{(1)}[x] = j_\ell[x] + in_\ell[x], \tag{4.51}$$

$$h_\ell^{(2)}[x] = j_\ell[x] - in_\ell[x]. \tag{4.52}$$

In this chapter, we developed three techniques for solving differential equations. The first of these techniques is separation of variables, which we will in use Chapter 5 in order to turn the partial differential equation Eqn. 3.62, our unified wave equation, into three ordinary differential equations. The remaining two techniques, infinite series solutions and associated function solutions, will then be used to solve two of the three ordinary differential equations that will result from the separation of Eqn. 3.62.

Chapter 5

Solutions to Unified Wave Equation

Now we are ready to solve Eqn. 3.62, our unified wave equation, for the objective function ψ_i,

$$\nabla^2 \psi_i - k_i^2 \psi_i = 0.$$

We will begin by substituting Eqn. 2.92, the Laplacian in spherical coordinates,

$$\begin{aligned} 0 = &\frac{1}{r^2}\partial_r\left(r^2 \partial_r \psi_i\left[r,\theta,\phi\right]\right) + \frac{1}{r^2 \sin\left[\theta\right]}\partial_\theta\left(\sin\left[\theta\right]\partial_\theta \psi_i\left[r,\theta,\phi\right]\right) \\ &+ \frac{1}{r^2 \sin^2\left[\theta\right]}\partial_\varphi^2 \psi_i\left[r,\theta,\phi\right] - k_i^2\left[r\right]\psi_i\left[r,\theta,\phi\right]. \end{aligned} \tag{5.1}$$

Note that we have written the functional dependence of ψ_i and k_i. Next we will apply separation of variables in order to solve Eqn. 5.1.

5.1 Applying Separation of Variables

First, we will assume a product solution for ψ_i. Let $\psi_i = R_i[r]\,Y_i[\theta,\varphi]$. Making this substitution gives us,

$$0 = \frac{1}{r^2}\partial_r\left(r^2\partial_r R_i[r]\,Y_i[\theta,\varphi]\right) + \frac{1}{r^2\sin[\theta]}\partial_\theta\left(\sin[\theta]\,\partial_\theta R_i[r]\,Y_i[\theta,\varphi]\right) + \frac{1}{r^2\sin^2[\theta]}\partial_\varphi^2 R_i[r]\,Y_i[\theta,\varphi] - k_i^2[r]\,R_i[r]\,Y_i[\theta,\varphi]. \tag{5.2}$$

Since R_i only depends on r, R_i is constant with respect to θ and φ derivatives. Similarly, since Y_i only depends on θ and φ, Y_i is constant with respect to r derivatives. Taking advantage of these facts we can simplify our differential equation,

$$0 = \frac{Y_i[\theta,\varphi]}{r^2}\partial_r\left(r^2\partial_r R_i[r]\right) + \frac{R_i[r]}{r^2\sin[\theta]}\partial_\theta\left(\sin[\theta]\,\partial_\theta Y_i[\theta,\varphi]\right) + \frac{R_i[r]}{r^2\sin^2[\theta]}\partial_\varphi^2 Y_i[\theta,\varphi] - k_i^2[r]\,R_i[r]\,Y_i[\theta,\varphi]. \tag{5.3}$$

Now we will multiply our equation by $r^2/(R_iY_i)$ and simplify,

$$0 = \frac{1}{R_i[r]}\partial_r\left(r^2\partial_r R_i[r]\right) + \frac{1}{\sin[\theta]\,Y_i[\theta,\varphi]}\partial_\theta\left(\sin[\theta]\,\partial_\theta Y_i[\theta,\varphi]\right) + \frac{1}{\sin^2[\theta]\,Y_i[\theta,\varphi]}\partial_\varphi^2 Y_i[\theta,\varphi] - r^2k_i^2[r]. \tag{5.4}$$

Notice that we can collect terms that only depend on r and terms that are only dependent on θ and φ. Hence, we can perform separation of variables on this differential equation. In Section 4.1, we derived the separation constant to have a completely arbitrary form λ. However, in some differential equations it

is useful to assume a particular form for the separation constant. In this case, we will choose $\ell(\ell+1)$ as the separation constant,

$$\ell(\ell+1) = \frac{1}{R_i[r]}\partial_r\left(r^2\partial_r R_i[r]\right) - r^2k_i^2[r], \tag{5.5}$$

$$-\ell(\ell+1) = \frac{1}{\sin[\theta]\,Y_i[\theta,\varphi]}\partial_\theta\left(\sin[\theta]\,\partial_\theta Y_i[\theta,\varphi]\right) + \frac{1}{\sin^2[\theta]\,Y_i[\theta,\varphi]}\partial_\varphi^2 Y_i[\theta,\varphi]. \tag{5.6}$$

where ℓ is a nonnegative integer. The motivation of why we choose a separation constant of the form $\ell(\ell+1)$ can be found in [1]. Now, we have an ordinary differential equation in r for the objective function R_i and a partial differential equation in θ and φ for the objective function Y_i.

Note that the Y_i differential equation contains no terms that are different between electrons and photons, as such we will drop the i subscript and simply call the objective function Y. We will perform separation of variables again in order to separate Y into two ordinary differential equations. Multiplying Eqn. 5.6 by $\sin^2[\theta]\,Y$ yields,

$$-\ell(\ell+1)\sin^2[\theta]\,Y = \sin[\theta]\,\partial_\theta\left(\sin[\theta]\,\partial_\theta Y[\theta,\varphi]\right) + \partial_\varphi^2 Y[\theta,\varphi]. \tag{5.7}$$

Now we will assume a product solution, $Y[\theta,\varphi] = \Theta[\theta]\,\Phi[\varphi]$. After substituting in our product solution into Eqn. 5.7, then multiplying by $1/(\Theta\Phi)$ and simplifying we have,

$$0 = \frac{1}{\Theta[\theta]}\sin[\theta]\,\partial_\theta\left(\sin[\theta]\,\partial_\theta\Theta[\theta]\right) + \ell(\ell+1)\sin^2[\theta] + \frac{1}{\Phi[\varphi]}\partial_\varphi^2\Phi[\varphi]. \tag{5.8}$$

Since we can collect terms that only depend on θ and terms that only depend on φ, we can separate variables. We will choose m^2 as the separation constant,

$$m^2 = \frac{1}{\Theta[\theta]} \sin[\theta] \partial_\theta (\sin[\theta] \partial_\theta \Theta[\theta]) + \ell(\ell+1) \sin^2[\theta] \tag{5.9}$$

$$-m^2 = \frac{1}{\Phi[\varphi]} \partial_\varphi^2 \Phi[\varphi]. \tag{5.10}$$

Once again, the motivation of why we choose a separation constant of the form m^2 is shown in [1].

5.2 Solutions to the Angular Differential Equations

The solution to Eqn. 5.10 is easy; it is a linear combination of complex exponential functions,

$$\Phi[\varphi] = C_1 e^{im\varphi} + C_2 e^{-im\varphi}, \tag{5.11}$$

where C_1 and C_2 are arbitrary constants [4]. When solving the differential equation for Θ, we will allow m to be negative. If m is allowed to be negative, then $e^{im\varphi}$ and $e^{-im\varphi}$ are not linearly independent functions. Therefore, we can discard one of the two terms in Eqn. 5.11. We choose to discard the second term, setting $C_2 = 0$. We will absorb the arbitrary constant C_1 into the aribitrary constant for the solution to the differential equation for Θ. Thus, our final solution to the differential equation for Φ is given by,

$$\Phi[\varphi] = e^{im\varphi}, \tag{5.12}$$

Since φ is the azimuthal coordinate, it is reasonable to require that $\Phi[\varphi] = \Phi[\varphi + 2\pi]$. Applying this restriction results in $e^{i2\pi m} = 1$, which implies that $m \in \mathbb{Z}$.

Eqn. 5.9, the differential equation for Θ, is the *associated Legendre differential equation*,

$$\left(1 - x^2\right)\partial_x^2 y - 2x\partial_x y + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)y = 0, \tag{5.13}$$

with the object function y being Θ and the independent variable $x = \cos[\theta]$. The solutions to the Eqn. 5.9 are of the form,

$$\Theta[\theta] = C_\ell P_\ell^m[\cos[\theta]], \tag{5.14}$$

where C_ℓ is a constant, and P_ℓ^m is the associated Legendre function, which is given by,

$$P_\ell^m[x] = \left(1 - x^2\right)^{|m|/2} \partial_x^{|m|} P_\ell[\cos[\theta]], \tag{5.15}$$

where P_l is the Legendre polynomial with index ℓ [4]. Recall that the Legendre polynomials are given by Rodrigues's formula,

$$P_\ell = \frac{1}{2^\ell \ell!}\partial_x^\ell\left(x^2 - 1\right)^\ell.$$

The degree of $(x^2-1)^\ell$ is 2ℓ. Since we apply ℓ derivatives to $(x^2-1)^\ell$ in order to obtain P_ℓ, the degree of P_ℓ equal to ℓ. Thus, if $|m| > \ell$, then Eqn. 5.15 will be zero; that is, we have an additional restriction that $-\ell \leq m \leq \ell$. Hence for every value of ℓ there are $2\ell + 1$ allowed values for m, as shown in [4]. Now we

can write our full angular equation,

$$Y_{\ell,m}[\theta,\varphi] = C_\ell P_\ell^m[\cos[\theta]]\, e^{im\varphi}. \tag{5.16}$$

5.3 Solutions to the Radial Differential Equation

Next, we will solve Eqn. 5.5, the differential equation for R_i. We must make a choice for $\chi[r]$ in order to solve for R_i. The choice of $\chi[r]$ represents the physics of the system that we are analyzing. We will look at a system in which the effective potential in constant inside a boundary at $r = a$ and is a different constant outside of that boundary. Thus we choose $\chi[r]$ to be piecewise constant,

$$\chi[r] = \begin{cases} 0 & r < a, \\ 1 & r \geq a. \end{cases} \tag{5.17}$$

Making this choice causes Eqn. 5.5 to reduce to the Spherical Bessel differential equation,

$$x^2\partial_x^2 y + 2x\partial_x y + \left(x^2 - \ell(\ell+1)\right) y = 0,$$

with objective function y being rR_i and independent variable $x = k_i r$. The solutions to this equation are the spherical Bessel functions of the first kind j_ℓ and spherical Bessel functions of the second kind n_ℓ as derived in Chapter 4. Since j_ℓ and n_ℓ are linearly independent, the general solution to Eqn. 5.5, the radial differential equation, is a linear combination of these functions,

$$rR_i = C_1 j_\ell[k_i r] + C_2 n_\ell[k_i r]. \tag{5.18}$$

Our full solution in terms of the Spherical Bessel functions is

$$\psi_i^{\ell,m}\left[r,\theta,\phi\right] = \left(C_1 j_\ell\left[k_i r\right] + C_2 n_\ell\left[k_i r\right]\right) C_\ell P_\ell^m\left[\cos\left[\theta\right]\right] e^{im\varphi} \tag{5.19}$$

$$= \left(A_\ell j_\ell\left[k_i r\right] + B_\ell n_\ell\left[k_i r\right]\right) P_\ell^m\left[\cos\left[\theta\right]\right] e^{im\varphi}, \tag{5.20}$$

where we have absorbed the indexed arbitrary constant C_ℓ into C_1 and C_2 and relabeled them A_ℓ and B_ℓ, respectively.

Now, we will use boundary conditions in order to determine the constants A_ℓ, B_ℓ, and k_i in Eqn. 5.20. Note that since the undetermined constants only appear in the radial part of the solution, we can simply work with the radial part of the equation,

$$rR_i = A_\ell j_\ell\left[k_i r\right] + B_\ell n_\ell\left[k_i r\right]. \tag{5.21}$$

Recall that our system involves a spherical boundary with a radius of a. This means that the wave function could be different both inside and outside of the boundary, as such we will let R_i^{in} be the wave function inside of the sphere and R_i^{out} be the wave function outside of the sphere,

$$rR_i^{in} = A_\ell^{in} j_\ell\left[k_i^{in} r\right] + B_\ell^{in} n_\ell\left[k_i^{in} r\right]. \tag{5.22}$$

$$rR_i^{out} = A_\ell^{out} j_\ell\left[k_i^{out} r\right] + B_\ell^{out} n_\ell\left[k_i^{out} r\right]. \tag{5.23}$$

Our first boundary condition is that we will require that the inside wave function be finite as the radius goes to zero. Bessel functions of the second kind have a vertical asymptote at the origin, thus $B_\ell^{in} = 0$ for all $\ell \in \mathbb{N}$ and the

radial equation inside the sphere becomes,

$$rR_i^{in}\left[k_i^{in}r\right] = A_\ell^{in} j_\ell\left[k_i^{in}r\right]. \tag{5.24}$$

In order to simplify R_i^{out}, we will first re-express R_i^{out} in terms of an equivalent basis, the Spherical Hankel functions,

$$rR_i^{out}\left[k_i^{out}r\right] = C_\ell^{out} h_\ell^{(1)}\left[k_i^{out}r\right] + D_\ell^{out} h_\ell^{(2)}\left[k_i^{out}r\right]. \tag{5.25}$$

Note that the Spherical Hankel functions of the first kind correspond to waves propagating away from the origin and the Spherical Hankel functions of the second kind correspond to waves propagating away from the origin. We will not be considering a case in which waves are being introduced to the system from outside of the boundary, hence we can set $D_\ell^{out} = 0$,

$$rR_i^{out}\left[k_i^{out}r\right] = C_\ell^{out} h_\ell^{(1)}\left[k_i^{out}r\right]. \tag{5.26}$$

We will apply the usual quantum mechanical boundary conditions, requiring that the wave function be continuous at the boundary,

$$R_i^{in}\left[k_i^{in}a\right] = R_i^{out}\left[k_i^{out}a\right], \tag{5.27}$$

and be differentiable at the boundary,

$$\partial_r R_i^{in}\left[k_i^{in}r\right]\Big|_{r=a} = \partial_r R_i^{out}\left[k_i^{out}r\right]\Big|_{r=a}. \tag{5.28}$$

Note that the boundary conditions that we would normally apply when working with Maxwell's equations are fundamentally different from the quantum mechanical boundary conditions, partially due to the fact that electric fields (photon wave functions) are vector quantities and electron wave functions are scalar quantities. In a similar manner to proving the assertion that considering a scalar wave equation is sufficient to fully analyze the radial and temporal behavior of the electric field in Chapter 3, using ideas from [7], it can be shown that in our problem, the usual quantum mechanical boundary conditions and usual electrodynamic boundary conditions are logically equivalent. Next, we will substitute Eqn. 5.24 and Eqn. 5.26 into Eqn. 5.27,

$$A_\ell^{in} j_\ell \left[k_i^{in} a\right] = C_\ell^{out} h_\ell^{(1)} \left[k_i^{out} a\right]. \tag{5.29}$$

We also substitute Eqn. 5.24 and Eqn. 5.26 into Eqn. 5.28 and simplify to get

$$A_\ell^{in}\ \partial_r j_\ell \left[k_i^{in} r\right]\Big|_{r=a} = C_\ell^{out}\ \partial_r h_\ell^{(1)} \left[k_i^{out} r\right]\Big|_{r=a}. \tag{5.30}$$

Now, we can divide Eqn. 5.30 by Eqn. 5.29 and multiply by r to get

$$\left.\frac{k_i^{in} r \partial_r j_\ell \left[k_i^{in} r\right]}{j_\ell \left[k_i^{in} r\right]}\right|_{r=a} = \left.\frac{k_i^{out} r \partial_r h_\ell^{(1)} \left[k_i^{out} r\right]}{h_\ell^{(1)} \left[k_i^{out} r\right]}\right|_{r=a}. \tag{5.31}$$

Eqn. 5.31 is called the **characteristic equation** of our system. In order to begin solving the characteristic equation we will first recall Eqn. 3.61,

$$k_i^2 = -\frac{\alpha_i \Delta_i \chi\left[r\right]}{\lambda_i^2}\psi_i + \frac{1}{\lambda_{i0}^2},$$

and the definitions of the constants in Eqn. 3.61 for photons,

$$\alpha_\gamma = 1, \tag{5.32}$$

$$\Delta_\gamma = \frac{n^2[0] - n^2[a]}{n^2[0]}, \tag{5.33}$$

$$\lambda_i^2 = \lambda_{i0}^2 = \frac{c^2}{\omega^2 n^2[0]}. \tag{5.34}$$

Substituting Eqn. 5.32, Eqn. 5.33, and Eqn. 5.34 into Eqn. 3.61 and resimplifying yeilds,

$$k_\gamma^2 = \frac{\omega^2 n^2[0]}{c^2}\left(1 - \Delta_\gamma \chi[r]\right). \tag{5.35}$$

Recall Eqn. 5.17 and note that we are considering the case in which $\chi[r]$ is a piecewise function with a discontinuity at the boundary,

$$\chi[r] = \begin{cases} 0 & r < a, \\ 1 & r \geq a. \end{cases}$$

By subsisting Eqn. 5.17 into Eqn. 5.35 and simplifying, we get expressions for k_γ^{in} and k_γ^{out},

$$k_\gamma^{in} = \frac{\omega^2 n^2[0]}{c^2} \tag{5.36}$$

$$k_\gamma^{out} = \frac{\omega^2 n^2[0]}{c^2}\sqrt{1 - \Delta_\gamma} = k_\gamma^{in}\sqrt{1 - \Delta_\gamma}. \tag{5.37}$$

In order to make the algebra simpler, we define new constants to indicate the

function arguments of the left and right hand side of our expression,

$$u_{in} = k_\gamma^{in} a = \frac{\omega^2 n^2 [0]}{c^2} a, \tag{5.38}$$

$$u_{out} = k_\gamma^{out} a = u_{in} \sqrt{1 - \Delta_\gamma}. \tag{5.39}$$

Hence, our characteristic equation becomes,

$$\frac{u_{in} \partial_r j_\ell [u_{in}]}{j_\ell [u_{in}]} = \frac{u_{in} \sqrt{1 - \Delta_\gamma} \partial_r h_\ell^{(1)} \left[u_{in} \sqrt{1 - \Delta_\gamma} \right]}{h_\ell^{(1)} \left[u_{in} \sqrt{1 - \Delta_\gamma} \right]}, \tag{5.40}$$

where the notation $\partial_r j_\ell [u_{in}]$ means to take the r derivative of j_ℓ, then evaluate the result at u_{in}, and so forth. So we have an equation with one unknown u_{in}. Unfortunately, this equation is not analytically solvable for u_{in}. However, we can numerically solve for u_{in}.

5.4 Numerical Solutions to the Characteristic Equation

We derived the Spherical Bessel functions in terms of the normal Bessel functions. When seeking numerical solutions to the characteristic equation, it will be useful to use the following generating function for the Spherical Bessel function of the first kind [4],

$$j_\ell [x] = (-x)^\ell \left(\frac{1}{x} \partial_x \right)^\ell \left(\frac{\sin [x]}{x} \right).$$

There are actually a countably infinite number of characteristic equations, one for each value of ℓ. Also, for each choice of the "quantum number" ℓ there are a countably infinite number of solutions for u_{in}. We speak of quantum numbers in quotations because we are not truly applying quantum mechanics to the photon case, but we are borrowing ideas from quantum mechanics to analyze the photon case. We label the solutions for a particular ℓ with a new index, the *order* n. We also refer to n as the *radial quantum number*. The $n = 1$ solution corresponds to the smallest value of u_{in} for which the characteristic equation is satisfied and the higher order solutions are labeled in ascending order. Since we have an infinite number of possible numerical solutions, we will only be examining solutions for select values of ℓ and n.

In order to use numerical methods to approximate solutions to the characteristic equation, we must first choose values for the physical parameters of the problem. We will choose the radius of the sphere to be 1 μm and set the index of refraction of the inside of the sphere to 1.5 and outside of the sphere to 1. Making these substitutions, for this particular example we have

$$\Delta_\gamma = 5/9, \tag{5.41}$$

$$u_{out} = \frac{2}{3} u_{in}. \tag{5.42}$$

Now, we will use numerical solving techniques in Mathematica to obtain numerical solutions to Eqn. 5.40, the characteristic equation. We will visualize these solutions in two ways. First, we will plot the numerical solution for R^2

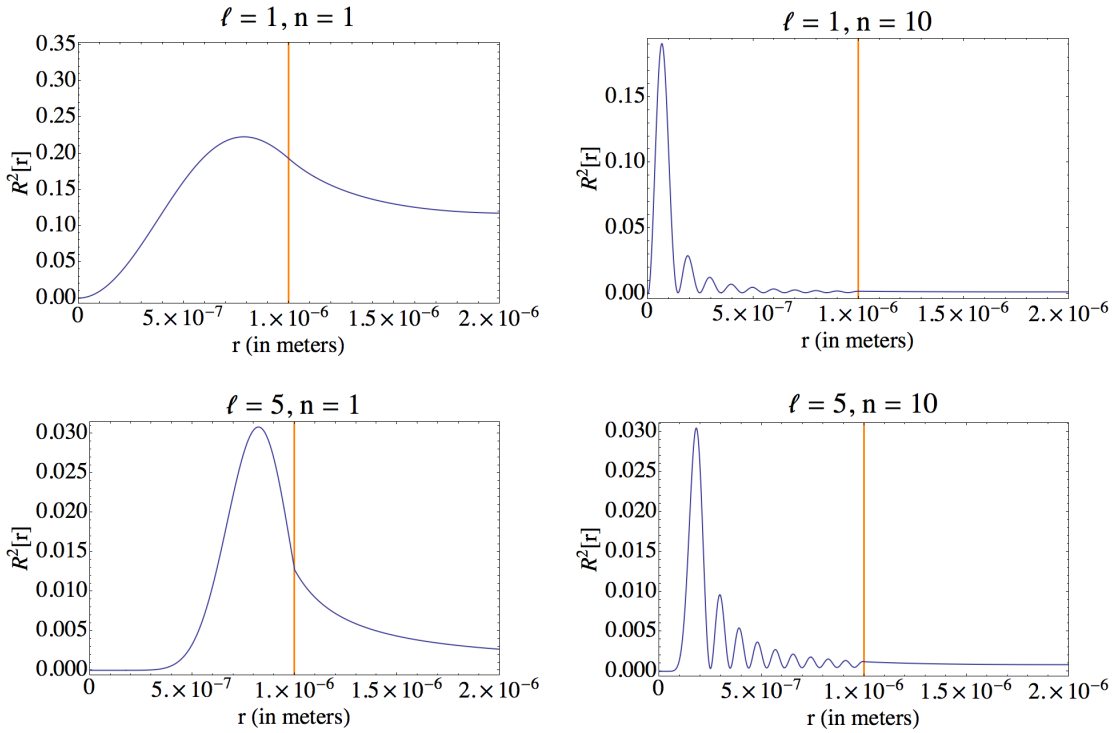

Figure 5.1: Several Plots of the Radial Part R^2 of the Wave Function ψ versus the radial coordinate r

versus r for particular values of ℓ and n. A sample of four plots, spanning two different angular momentum values $\ell \in \{1, 5\}$ and two different orders $n \in \{1, 10\}$ are shown in Fig. 5.1. Remember that peaks in $\mathbf{E}^2$, and therefore peaks in R^2, correspond to points of maximal intensity for the light. Notice that as the angular momentum quantum number increases, the peaks inside the boundary bunch up closer to the boundary. If you consider the nature of the centrifugal pseudo-force on a classical object, this makes sense intuitively. As we increase the velocity of an orbiting classical object inside a sphere, it can orbit further away from the origin, going against the centrifugal pseudo-force. This is analogous to increasing the angular momentum quantum number of

light causing the light to be more intense closer the boundary of the sphere. It is also important to note that as the radial quantum number increases, the number of peaks that there are inside the boundary also increases. Once again, this makes sense in the classical analogue. If we impart more radial momentum on an object, then it will have more motion along its radial axis.

The second key feature of the numerical solutions that we will be analyzing is the temporal behavior of the wave function. When first solving our general differential equation, we immediately solved for the temporal part of our equation,

$$\psi_i[r,\theta,\varphi,t] = \psi_i[r,\theta,\varphi]\, e^{-i\omega t}. \tag{5.43}$$

The temporal frequency ω is simply the spatial frequency k multiplied by the speed of light in the medium, c/n_{in}. But k is complex in general, so Eqn. 5.43 becomes

$$\psi_i[r,\theta,\varphi,t] = \psi_i[r,\theta,\varphi]\, e^{(ck_{im}t)/n_{in}} e^{(-ick_{re}t)/n_{in}}, \tag{5.44}$$

where k_{re} and k_{im} are the real and imaginary parts of k, respectively. Eqn. 5.44 demonstrates the solutions for the wave number inside the spherical boundary k, in order to obtain the wave number for outside the boundary k^{out}, we require continuity of the wave in time. This requires that $\omega_{in} = \omega_{out}$,

$$\frac{k^{out}c}{n^{out}} = \frac{k^{in}c}{n^{in}} \tag{5.45}$$

$$k^{out} = \frac{n^{out}}{n^{in}}k^{in}. \tag{5.46}$$

Hence, we will only be examining time constants corresponding to k^{in} since the time constants corresponding to k^{out} will just be a positive constant

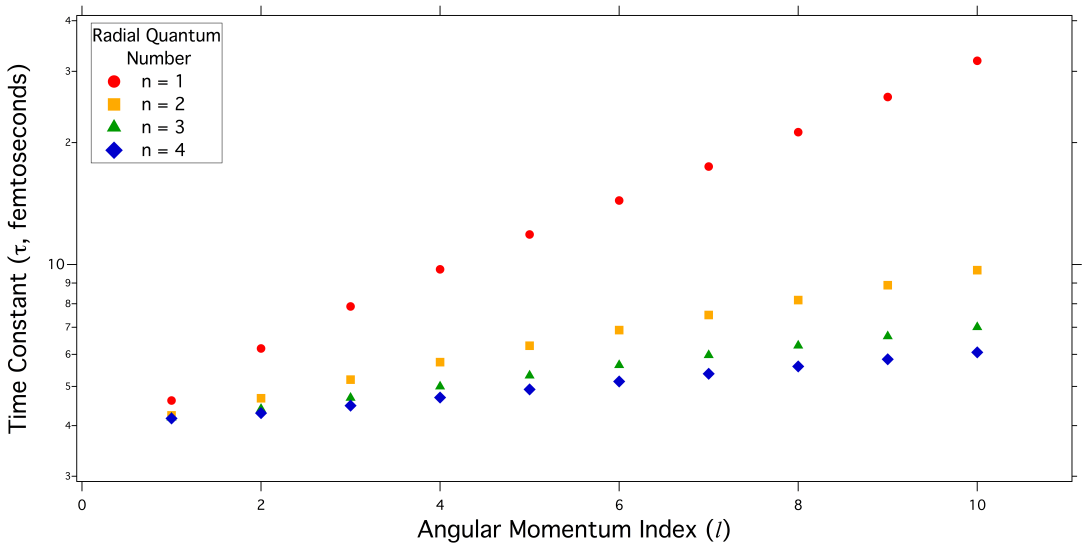

Figure 5.2: Logarithm of Time Constant vs Angular Momentum for the First Four Solution Orders

multiple of the time constants for k^{in}. Note that for electrons k turns out to be real, so we do not see amplitude decay in electrons [4]. It is also important to note that for all of the choices of ℓ and n that we have computed numerically, k_{im}^{in} is negative. Therefore, our amplitude will decay in time. In particular, we will define the time constant τ as the time at which the amplitude of the radial function has decayed by $1/e$,

$$\tau = -\frac{1}{ck_{im}}. \tag{5.47}$$

Just as with the plots of R versus r, we are going to visualize the time constants in such a way that we see the behavior of the time constants as we vary angular momentum and radial solution order. In Fig 5.2, we have the logarithm of the time constant τ versus the angular momentum from $\ell = 1$ to $\ell = 10$ for the first four orders of solutions $n = 1$ to $n = 4$. This graph highlights

two key relationships: as the angular momentum quantum number increases, the longevity of the solution increases, but as the radial quantum number increases, the longevity of the solution decreases. Both of these relationships are physically logical in a classical analogue. A rotating system with more angular momentum will spin for longer periods of time and is more likely to stay in a configuration that is spinning. This corresponds to light staying inside the sphere longer. A classical rotating system with more radial momentum will not spin for as long because it has more energy along the radial direction, which is the direction orthogonal to the rotation. This corresponds to light leaving the sphere. For the shown values of n and ℓ, the time constants were on the order of femtoseconds.

Another way that we can analyze the temporal decay of the solutions is to look at the ratio of the time constants to the period of oscillation. That is, we will examine how many times the solution goes through a full period of oscillation before the amplitude of the solutions decays by a factor of $1/e$. In order to do so, we will compute the period of oscillation T of the system. The frequency of oscillation f is given by $f = \omega_{re}^{in}/(2\pi) = ck_{re}^{in}/(2\pi n^{in})$. The period of oscillation is the reciprocal of the frequency, hence

$$T = \frac{2\pi n^{in}}{ck_{re}^{in}}. \tag{5.48}$$

We can compute the ratio of the time constant to the period of oscillation by

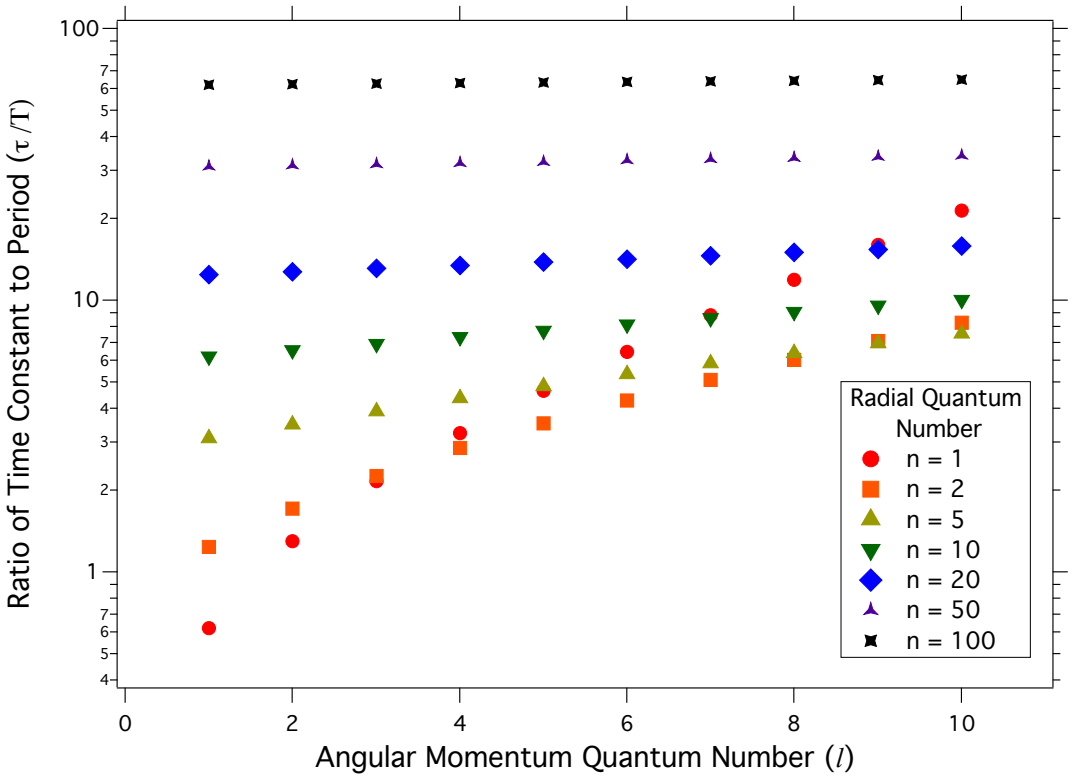

Figure 5.3: Logarithm of Ratio of Time Constant to Period of Oscillation vs Angular Momentum Quantum Number for several choices of radial quantum number.

dividing Eqn. 5.47 by Eqn. 5.48,

$$\frac{\tau}{T} = -\frac{k_{re}^{in}}{2\pi k_{im}^{in}}. \tag{5.49}$$

In Fig. 5.3 we plotted τ/T versus the angular momentum quantum number for several choices radial quantum in several orders of magnitude, $n \in \{1, 2, 5, 10, 20, 50, 100\}$. An interesting feature of the solutions that this plot illustrates is that increasing the angular momentum quantum number has a much more dramatic effect on τ/T for lower radial quantum number solutions than higher radial quantum number solutions. For example, cosider the range

of $\ell = 1$ to $\ell = 10$. For $n = 1$ τ/T, increases from 0.621 to 21.4. But, for $n = 100$ τ/T, increases from 62.1 to 65.1 on the same interval. So by increasing the angular momentum quantum number from $\ell = 1$ to $\ell = 10$, the $n = 1$ solution will have 34.4 times more oscillations before its amplitude decays by a factor of $1/e$, but the $n = 100$ solution only sees an 1.05 times more oscillations with the same increase in angular momentum. This idea makes sense intuitively. Consider a classical rotating system with a given value of angular momentum. As we increase the radial momentum of the system, the overall impact of imparting extra angular momentum on the system will decrease because the angular momentum constitutes a lower percentage of the system's total momentum.

Another feature of our solution set that Fig. 5.3 allows us to compare the number of oscillations before the amplitude decays by a factor of $1/e$ for various choices of radial quantum number for a particular angular momentum quantum number. For example, for $1 \leq \ell \leq 3$, the first radial quantum number solution (n=1) has the least amount of cycles before the amplitude decays by $1/e$. But at $\ell = 4$, the $n = 2$ solution has less cycles before amplitude decays by a factor of $1/e$ than the $n = 1$ solution. In general as the angular momentum quantum number increases, the radial quantum number with the least amount of cycles before the amplitude decays by a factor of $1/e$ increases.

We can also compute the wavelength of light that will be bound in the sphere for a given value of n and ℓ. The wavelength will be the product of the

speed of the wave and its period,

$$\lambda^{in} = v^{in}T \tag{5.50}$$

$$= \left(\frac{c}{n^{in}}\right)\left(\frac{2\pi n^{in}}{ck_{re}^{in}}\right) \tag{5.51}$$

$$= \frac{2\pi}{k_{re}^{in}}. \tag{5.52}$$

For the $(n = 1, \ell = 1)$ solution, we have $k_{re}^{in} = 2.8 \cdot 10^6 \text{ m}^{-1}$. Therefore the wavelength of light inside the sphere is approximately 2200 nm. We could determine the wavelength of light in a vacuum that would be needed to excite the the $(n = 1, \ell = 1)$ solution λ_{exc},

$$\lambda_{exc} = \lambda^{in} n^{in}. \tag{5.53}$$

With $n^{in} = 1.5$, we would need a 3300 nm light source to excite the $(n = 1, \ell = 1)$ solution.

In this chapter, we solved the unified differential equation and analyzed several solutions for the photon case. We analyzed both the radial and temporal behavior of several numerical solutions and found behavior that follows intuition based on Newtonian dynamics.

Chapter 6

Conclusion and Future Work

Electrons and photons are fundamentally distinct objects with striking similarities. We have shown that the differential wave equations for electrons and photons in a spherically symmetric potential can be expressed in a compact unified form,

$$\left(\nabla^2 - k^2 + H'\right)\psi = 0. \tag{6.1}$$

The solutions to this differential equation are different for electrons and photons, the primary feature of this difference being that the wave number k must be real for electrons and can be complex for photons. The electron solution is well known as the three-dimensional finite spherical well problem in quantum mechanics [4].

We determined equivalent solutions for the photon case that had rich and intuitive dynamics. We found that the radial part of the photon wave function exhibited behavior that would be expected of a classical orbiting object. As we increase the angular momentum of the light wave, the peak intensity of the

light shifted further away from origin. Increasing the radial momentum of the light caused the light wave to oscillate more times inside the sphere. For photons, the wave number was found to be complex, while electrons have a real wave number. This meant that the temporal part of the photon wave function behaved like a dampened oscillator. The longevity of the solution increased as radial momentum decreased and as angular momentum increased. This makes intuitive sense as well, an orbiting object will stay in its orbit for longer if given more angular momentum. An orbiting object whose radial momentum is increasing will become increasingly more likely to leave its orbit.

There are several directions in which this project could continue. One such avenue is exploring the corrective term H'. The form of this term appears to be different for photons and electrons. However, as demonstrated by our work with algebraically manipulating k, there may be a sequence of substitutions and relabelling that would result in in these terms looking analogous. Another path that could be explored is other choices of χ, in particular choosing χ to have an inverse square dependence on r. The photon solution for this χ would correspond to an electron in a hydrogen atom.

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