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I Don't Play Chess: A Study of Chess Piece Generating Polynomials

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I Don't Play Chess: A Study of Chess Piece Generating Polynomials

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Abstract

This independent study examines counting problems of non-attacking rook, and non-attacking bishop placements. We examine boards for rook and bishop placement with restricted positions and varied dimensions. In this investigation, we discuss the general formula of a generating function for unrestricted, square bishop boards that relies on the Stirling numbers of the second kind. We discuss the maximum number of bishops we can place on a rectangular board, as well as a brief investigation of non-attacking rook placements on three-dimensional boards, drawing a connection to latin squares.

Dedication

To my parents,

for all of their many sacrifices to give me an education.

This was only possible because of you.

Acknowledgements

I would like to take this time to thank my advisor, Dr. Moynihan, for his patience, encouragement, and guidance throughout the process of writing this independent study. I cannot express my gratitude for all of your help and belief in my abilities. Thank you to Dr. Pierce, who gave me so much guidance and support throughout my years at the college. I would also like to thank the entire Mathematics department at The College of Wooster for being so helpful and approachable. In addition, I would like to thank all of the wonderful friends I have made at Wooster who have always been there for me. You have all taught me so much and kept me grounded. I am so grateful for each and every memory, and I am looking forward to seeing all of you change the world. Most importantly, I would like to thank my parents for being a constant source of love, support, and wisdom throughout every aspect of my life.

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Chapter 1

Introduction

> “During a heavy gale a chimney pot was hurled throught the air and crashed upon the pavement just in front of a pedestrian. He quite calmly said, ‘I have no use for it: I do not smoke.’ Some readers when they happen to see a puzzle represented on a chessboard with chess pieces, are apt to make the equally inconsequent remark, ‘I have no use for it: I do not play chess.’”
>
> - *Henry Ernest Dudeney* [4]

Chess is an amazingly complicated game with a seemingly infinite number of scenarios. The rules that govern the game of chess have proven to be an attractive area of inquiry to mathematicians the world over. Throughout the course of this independent study, we will take a look at some counting problems regarding rooks, bishops, and the chessboard itself.

For those unfamiliar with the game of chess, the rules are quite simple. The game is played on an 8×8 checkered board and two players take turns moving their pieces around the board. The objective is to "checkmate" the opponent's king, which means the king can be attacked on the next turn, and no matter how the opponent moves his pieces, there is no way to prevent the king from being attacked. The two players start with 8 pawns, 2 rooks, 2 knights, 2 bishops, 1 queen, and 1 king. These pieces differ only in the way they are allowed to move around the board and "attack" other pieces. For example, rooks can move and attack as many squares that are unoccupied along its row or column. Bishops can move and attack only along diagonals for as many squares that are unoccupied. The other pieces follow different rules for movement, but our investigation will focus on rooks and bishops. There are probably hundreds or thousands of books written on the strategy of chess. There are different scenarios that require intellegent decisions on how to move your pieces. We will not be examining any of these scenarious as the game of chess is not the focus of this study.

We will focus on counting problems that arise from the structures of chess pieces. As an example, think of a standard chessboard. What is the maximum number of rooks we can place on this board so that no rook is able to attack another? How many different ways can we place this maximum number of rooks so that they cannot attack each other? These questions leads us to other, more complicated questions. What if we asked the same questions, but instead of rooks, we want to know about bishops? What happens to these numbers when we change the dimensions of the board, or even add a dimension? The answers to these questions are not obvious and are usually difficult to compute.

We will strip away the game-like aspects from chess and mathematically examine some of the rules that govern chess pieces. Specifically, this independent study examines counting problems for non-attacking placements of both rooks and bishops on boards of various dimensions through the use of generating functions.

Chapter 2

Rook Boards of Two Dimensions

Let us begin by discussing the placement of non-capturing rooks in two dimensions. But first, a few definitions:

Definition 1. *A **board** is a grid for rook placement that has m rows and n columns for $m, n \in \mathbb{N}$.*

Definition 2. *A **non-capturing rook** or **non-attacking rook** is a rook placed on a board such that it shares no rows or columns with any other rook on the board.*

Because we want to place rooks on a board so that they are non-capturing, it will be safe to assume that unless otherwise specified from here on all rooks will be considered non-capturing. To illustrate this consider a standard 8×8 chessboard as shown in Figure 2.1.

Example 1. *How many different ways can we arrange 8 non-capturing rooks on a standard chessboard?*

To find the solution to this relatively easy problem let us look at the columns one at a time. Because there are 8 rooks, and none of them are

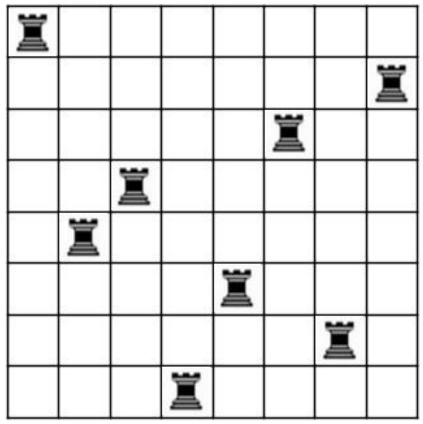

Figure 2.1: Eight non-capturing rooks on a standard chessboard.

allowed to share a column, then each column must have its own rook. In the first column we have 8 row choices to place a rook. Once this rook has been placed, we are not allowed to place another rook in the same row. When we move to the second column, we have 7 row choices to place our second rook. We can see by following this reasoning that 8! is the final number of ways to place 8 rooks on a chessboard. As you may note, this is also the number of ways to permute a set of 8 objects. It turns out you can think of a board as a representation of a permutation. Consider numbering the rows and columns each from 1 to n. If a rook is in row a and column b, then element b is permuted to element a. Because the rooks are non-attacking, there will be no double assigning of elements in the set.

When discussing non-attacking rook placements, it is important to remember that not all boards must be square as in the previous example. Because a board can be $m \times n$ it is fairly obvious that the maximum number of rooks we can place on the board is equal to the minimum of m and n. A simple calculation will yield that if we assume the minimum of m and n is m, then the number of ways to place the maximum number of rooks on a clear $m \times n$ board is

$$\binom{n}{m} \cdot m! = \frac{n!}{(n-m)!}.$$

A slightly more interesting counting problem is finding the number of ways to place k rooks on an $m \times n$ board. This leads us to Theorem 1, the proof for which draws from Feryal Alayont's article [1].

Theorem 1. *The number of ways to place k rooks on an unrestricted $m \times n$ board, where $0 \leq k \leq \min\{m, n\}$ is*

$$\binom{m}{k} \cdot \binom{n}{k} \cdot k!.$$

Proof. In order to place k non-attacking rooks on this unrestricted board, we need to choose k rows and k columns. Once we have selected these rows and columns we can begin by placing a rook in each column and row. Starting at the first column out of the k that we chose, we have k options to place a rook. Moving along to the second row, we have to remove one of the options because we have already placed a rook in one of the k rows. For our second column out of the k that we chose, we will only have $k-1$ options. Continuing along we see that the number of ways to place these rooks is $k!$. We multiply the number of ways to choose k rows, $\binom{m}{k}$, by the number of ways to choose k columns, $\binom{n}{k}$, and then again we multiply by the $k!$ number of ways to place rooks in the selected rows and columns. □

These blank boards are a somewhat uninteresting counting problem. Let us now start to restrict rooks so that they can only be placed in certain squares.

These squares that cannot hold rooks will be shown as blacked out squares. Keep in mind that while we are restricting where rooks can be placed, we are not restricting their ability to capture across these blacked out squares. We are still not allowed to have any rooks share a row or column. All boards are original unless otherwise specified.

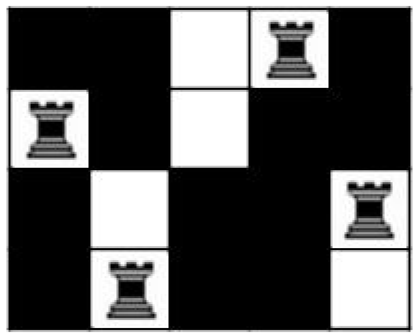

Figure 2.2: Non-capturing rooks on a 4×5 board with restricted squares.

Now that we have the basics of rooks and boards down, it is time to define the rook polynomial. We will look at the definition as formulated by Victor Bryant [3].

Definition 3. *Consider a board B of dimension $m \times n$. Without loss of generality assume that* $\min\{m, n\} = m$, *we define the* ***rook polynomial*** *of B to be*

$$R_B(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3 + \cdots + r_m x^m,$$

where r_k is the number of ways to place k non-attacking rooks on B.

It is important to note that in most cases the most interesting part of a rook polynomial is the leading coefficient. This coefficient represents the number of ways to place the maximal number of rooks on the board. It would also be appropriate to think of a rook polynomial as a generating function where the coefficient of x^k is the number of ways to place k rooks on the board. Like other generating functions, we are not interested in evaluating the function at any specific x value. We are more interested in the property that adding and multiplying polynomials will preserve the encoded information for our later theorems. As you will see later when we start multiplying rook polynomials, the way that they are added and multiplied will be able to tell us more information about more complicated boards.

Let B refer to the board in Figure 2.2. The rook polynomial is

$$R_B(x) = 1 + 8x + 21x^2 + 20x^3 + 6x^4.$$

It is somewhat easy to see that there are 8 ways to place 1 rook and 6 ways to place 4 rooks. However, even on a small board like this one, it is not obvious that there are 21 ways to place 2 rooks and 20 ways to place 3 rooks. On this small board it would not be too difficult to go through and count the different ways to place two or three rooks, but it is very inefficient. It should be noted that for any board $r_0 = 1$ as there is always only 1 way to place 0 rooks on any board.

Before we begin to discuss and prove some theorems that simplify the process of finding a rook polynomial for a given board, it is important to note that rearranging the board's rows and columns does not change the rook polynomial. All the squares in a given row will stay in that row, and all the squares in a given column will stay in the same column, so this will not alter the rook polynomial of any board [11].

Now let us look at some theorems that will simplify this exercise of finding the rook polynomial for a board. The following theorems and their proofs will borrow heavily from Victor Bryant's book [3].

Theorem 2. *Let B be a board which can be partitioned into two parts C and D which share no rows or columns. Then*

$$R_B(x) = R_C(x) \cdot R_D(x).$$

Proof. In order to prove this theorem, we must show that coefficients are the same on both sides of the equality. Because the boards are disjoint, placing a rook on C will not affect any rooks on D. The coefficient of x^k in $R_B(x)$ = (the number of ways to place 0 rooks on C and k on D) + (number of ways to place 1 rook on C and $k-1$ on D) + $\cdots$ + (number of ways to place k rooks on C and 0 on D).

This long description is the same as saying (coefficient of x^0 in $R_C(x)$) $\cdot$ (coefficient of x^k in $R_D(x)$) + (coefficient of x^1 in $R_C(x)$) $\cdot$ (coefficient of x^{k-1} in $R_D(x)$) + $\cdots$ + (coefficient of x^k in $R_C(x)$) $\cdot$ (coefficient of x^0 in $R_D(x)$). It is clear to see that this is equal to the coefficient of x^k in $R_C(x) \cdot R_D(x)$. □

Now let us consider our previous example in light of Theorem 2. If we swap columns 2 and 4 in the board from Figure 2.2, we get the board in Figure 2.3. We can clearly see the two sub-boards in Figure 2.4 These two boards by themselves are very simple and we can easily find the rook polynomials for the two boards by counting the number of ways to place rooks. We find that

$$R_{B_1}(x) = 1 + 4x + 3x^2$$

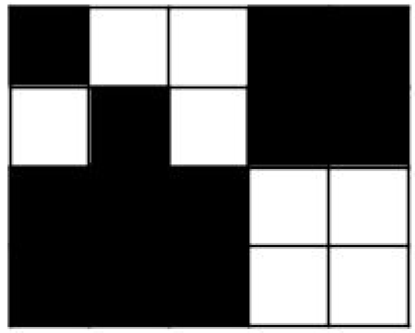

Figure 2.3: A board with two sub-boards sharing no rows or columns.

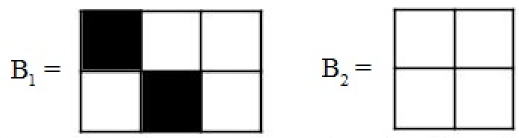

Figure 2.4: The board in Figure 2.3 separated into two disjoint sub-boards.

$$R_{B_2}(x) = 1 + 4x + 2x^2.$$

Also, we can clearly see that

$$R_{B_1}(x) \cdot R_{B_2}(x) = 1 + 8x + 21x^2 + 20x^3 + 6x^4 = R_B(x).$$

Before moving on to another theorem let us consider an example. The idea of a rook polynomial is simple enough, but what can this tell us? Consider a problem about finding the number of permutations of a set with restricted positions.

Example 2. *Consider permutations of the set* $\{1, 2, 3, 4\}$*. We restrict 1 to only map to 2, 2 to map to 1 or 2, 3 to map to 3 or 4, and 4 to map to either 3 or 4. How many permutations of this type are there?*

We could count each different permutation one by one, but it is much simpler to turn this into a rook problem. Let us construct a 4×4 board that will represent the restricted positions. If a cannot go to b we will block out (a, b) on the board as shown in Figure 2.5.

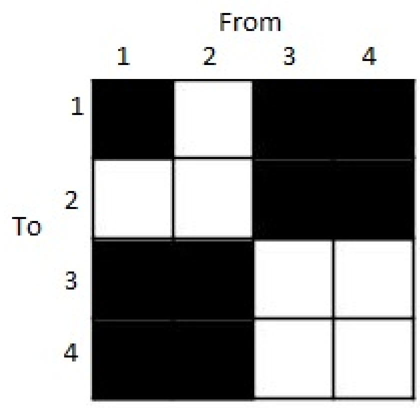

Figure 2.5: A board representing permutations with restricted positions.

Let P refer to the board in Example 2. Notice that P is clearly able to be partitioned into disjoint sub-boards. We will call the upper left board P_1 and the lower right one P_2. The rook polynomials are as follows:

$$R_{P_1}(x) = 1 + 3x + x^2$$
$$R_{P_2}(x) = 1 + 4x + 2x^2.$$

When we multiply them together, we can see the rook polynomial for P:

$$R_P(x) = 1 + 7x + 14x^2 + 10x^3 + 2x^4.$$

Remember that because of the way we defined this board, the coefficient of x^4 is exactly the number of ways to create permutations subject to the constraints of Example 2.

However, the diligent reader might notice that not every board will be separable. Some boards that are large and complicated might remain complicated even after swapping rows and columns. This observation leads us to our next theorem.

Theorem 3. *Let B be a board and let s be one particular square of that board. Then let B_1 be the board obtained from B by blacking out the square s and let B_2 be the board obtained from B by deleting the entire row and entire column containing s. Then*

$$R_B(x) = R_{B_1}(x) + x \cdot R_{B_2}(x).$$

Proof. In a similar fashion to the proof of Theorem 2, we need to show that every coefficient of x^k is the same on both sides of the equation. The coefficent of x^k in $R_B(x)$ = (Number of ways to place k rooks on B when a rook is not placed in s) + (Number of ways to place k rooks on B when a rook is placed in s) = (Number of ways to place k rooks on B_1) + (Number of ways to place $k-1$ rooks on B_2) = (coefficient of x^k in $R_{B_1}(x)$) + (coefficient of x^{k-1} in $R_{B_2}(x)$) = (coefficient of x^k in $R_{B_1}(x) + x \cdot R_{B_2}(x)$). Thus we can see that the coefficients are the same on both sides for every k. □

This theorem decomposes large board problems into many easier problems. By blacking out squares and deleting rows and columns, we get smaller boards. With this theorem we can continue to break down complicated boards into many smaller boards which have known rook polynomials. Eventually we will terminate with some separable boards that are small and have known rook polynomials. By Theorem 2, we can just multiply all their rook polynomials together to get the polynomial for the large board.

Example 3. *Find the rook polynomial of the board in the top left of Figure 2.6 using Theorem 3.*

As seen in Figure 2.6, we are able to break down the original board into two different kinds of sub-boards. Either we can find a small board where the rook polynomial is easy to calculate, or we can find a separable board where the rook polynomial is easy to find. If at any point in this process, the rook polynomial is not obvious for any board, we can apply Theorem 3 again so that we end up with even smaller boards. To find the solution to the problem given in Example 3, we must simply add together all the polynomials found. In Figure 2.6 the final rook polynomial for the top left board is

$$\begin{aligned} R_B(x) &= x(1+x)(1+2x) + x(1+2x)^2 + (1+3x+x^2)^2 \\ &= 1 + 8x + 18x^2 + 12x^3 + x^4. \end{aligned}$$

While these theorems allow us to calculate the rook polynomial for any board, as boards get large, this may not be the most efficient approach. If we had a very large board that was not separable, using Theorem 3 to find the rook polynomial might take a very long time. Now let us consider a very interesting observation that would not be obvious at first glance.

Definition 4. *The* ***complement*** *of a board B is the board where all squares of B that are restricted become open, and all open squares become restricted. We denote the complement of B as $\overline{B}$.*

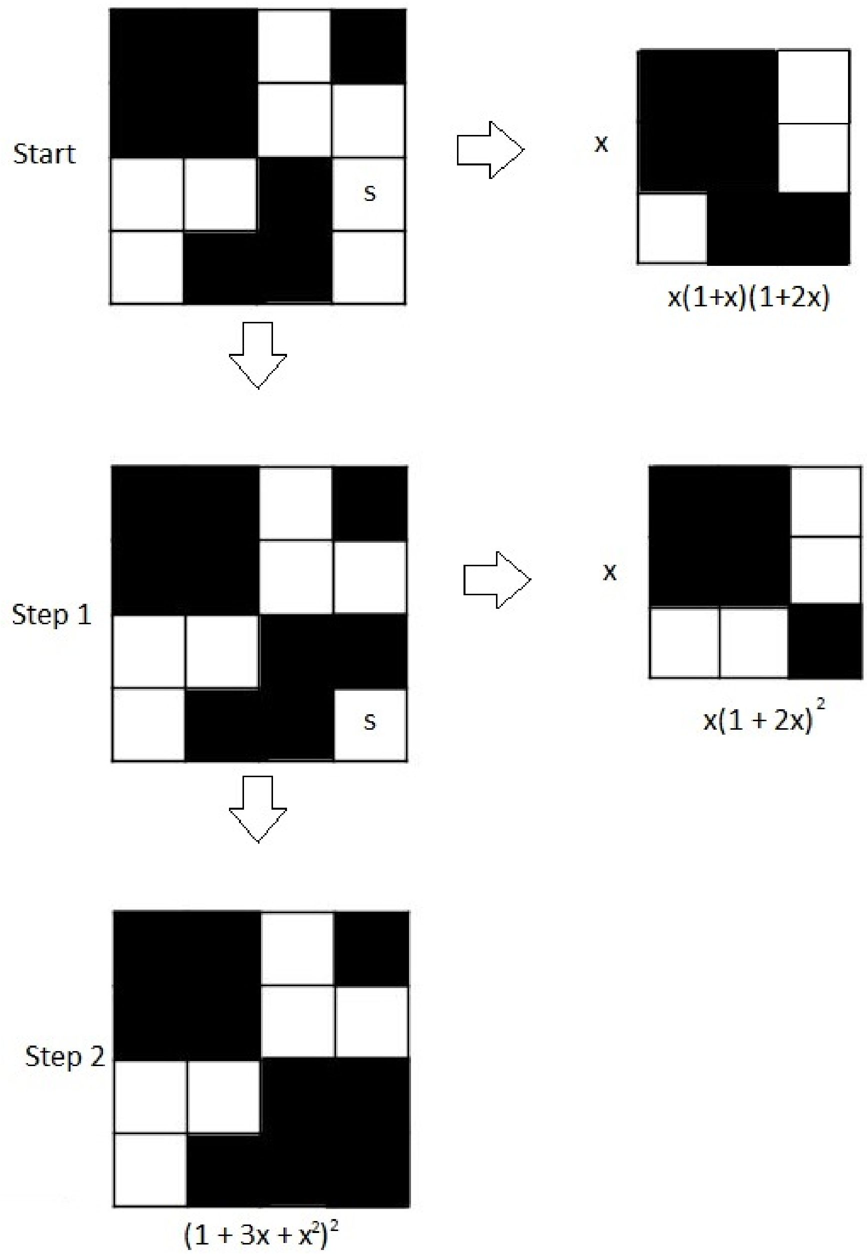

Figure 2.6: The decomposing of a non-separable board into boards with known rook polynomials.

Let us now look at a board and its complement. For the boards in

Figure 2.7 we have the following rook polynomials:

$$R_B(x) = 1 + 14x + 64x^2 + 112x^3 + 68x^4 + 9x^5$$

$$R_{\overline{B}}(x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 3x^5.$$

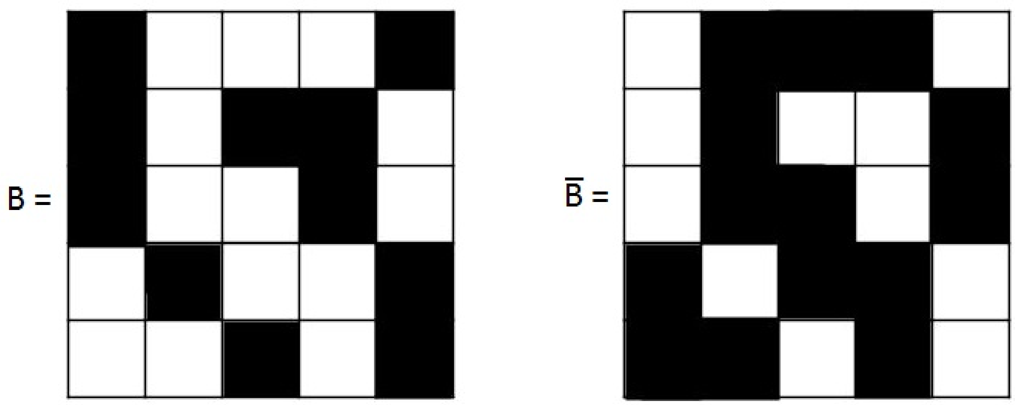

Figure 2.7: A board B and its complement $\overline{B}$.

These rook polynomials seemingly have no similarities at all. However, there is a relationship between these two sets of coefficients that is not obvious. Consider the coefficients of the rook polynomia $R_B(x)$.

$$\begin{aligned}(5! \cdot r_0) - (4! \cdot r_1) + (3! \cdot r_2) - (2! \cdot r_3) + (1! \cdot r_4) - (0! \cdot r_5) \\ = (5! \cdot 1) - (4! \cdot 14) + (3! \cdot 64) - (2! \cdot 112) + (1! \cdot 68) - (0! \cdot 9) \\ = 120 - 336 + 384 - 224 + 68 - 9 \\ = 3 = \text{Coefficient of } x^5 \text{ in } R_{\overline{B}}(x).\end{aligned}$$

Let's try this the other way now for $R_{\overline{B}}(x)$.

$$
\begin{aligned}
&(5! \cdot r_0) - (4! \cdot r_1) + (3! \cdot r_2) - (2! \cdot r_3) + (1! \cdot r_4) - (0! \cdot r_5) \\
&= (5! \cdot 1) - (4! \cdot 11) + (3! \cdot 40) - (2! \cdot 56) + (1! \cdot 28) - (0! \cdot 3) \\
&= 120 - 264 + 240 - 112 + 28 - 3 \\
&= 9 = \text{Coefficient of } x^5 \text{ in } R_B(x)
\end{aligned}
$$

It turns out that this relationship is no coincidence and it brings us to our next theorem.

Theorem 4. *Let B be a sub-board of an $n \times n$ board with rook polynomial*

$$R_B(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n,$$

and let $\overline{B}$ be the complement of B in the $n \times n$ board. Then the number of ways to place n non-challenging rooks on $\overline{B}$ equals

$$n!r_0 - (n-1)!r_1 + (n-2)!r_2 - \cdots + (-1)^n 0!r_n.$$

The following proof assumes the reader is familiar with the Inclusion-Exclusion Principle. Because we are looking to place n rooks on an $n \times n$ board, we must note that there will be exactly one rook in each column. However, as B is only a sub-board, the number of ways we can do this could be 0. This theorem will work for any board B but in order to apply the Inclusion-Exclusion Principle effectively, we must consider it as part of a larger board if it is not square. The reason we need to have a square board is we are going to be placing rooks in each row and column, so we must have an equal amount of rows and columns for this to work. We add rows or columns that are all blacked out squares in order to make it square. Since we are adding no possible places to put rooks the polynomial will not change. We will explain later that it is not necessary to black out these new rows. It should also be noted that when the proof refers to placing rooks “freely”, it means they could be placed temporarily on blacked out squares. When the word “freely” is used, any square is able to hold a rook. While this is not typically legal, the Inclusion-Exclusion Principle will eventually discount these possibilities by removing the intersections that contain these illegal rook placements.

Proof of Theorem 4. We defined B as a sub-board of an $n \times n$ board. Because our theorem only discusses placing n rooks onto $\overline{B}$ we must have one rook in each row and column of our larger $n \times n$ board. When we talk about rook 1, we are not numbering the rooks, we are referring to the rook in the first column. Define a function N in the folowing way: $N(1)$ = the number of ways to place the rook in the first column into B while placing the other rooks freely. Similarly $N(2)$ = the number of ways to place the rook in the second column into B while placing the other rooks freely. As a further example of the function N, observe that $N(1,2,3)$ = the number of ways to place the rooks in the first three columns into B while placing the other rooks freely. Say we have $N(i_1, i_2, ..., i_r)$ = (The number of ways to place the r rooks into B in their respective columns) $\cdot$ (number of ways to place the other $n-r$ rooks freely).

Now that we have defined N, we use the Inclusion-Exclusion Principle to calculate the number of ways to place n rooks on $\overline{B}$.

$$\begin{aligned}\text{Coefficient of } x^n \text{ in } R_{\overline{B}}(x) &= n! - N(1) - N(2) - \cdots - N(n)\\ &+ N(1,2) + N(1,3) + \cdots + N(n-1,n)\\ &- N(1,2,3) - N(1,2,4) - \cdots - N(n-2,n-1,n)\\ &\vdots\\ &+ (-1)^n N(1,2,...,n)\\ &= \sum_{r=0}^{n} \sum_{1\le i_1<i_2<...<i_r\le n} (-1)^r N(i_1,i_2,...,i_r). \qquad (1)\end{aligned}$$

Now that we have this coefficient of x^n in the rook polynomial of $\overline{B}$, we must compute $N(i_1, i_2, ..., i_r)$. As an example, let us consider $N(1,2,3)$.

$$\begin{aligned}&\sum_{1\le i_1<i_2<i_3\le n} (-1)^3 N(i_1,i_2,i_3)\\ &= N(1,2,3) + N(1,2,4) + ... + N(n-2,n-1,n)\\ &= (n-3)! \cdot (\text{Number of ways to place rooks 1,2,3 into } B)\\ &+ (n-3)! \cdot (\text{Number of ways to place rooks 1,2,4 into } B)\\ &\vdots\\ &+ (n-3)! \cdot (\text{Number of ways to place rooks } n-2, n-1, n \text{ into } B)\\ &= (n-3)! \cdot (\text{Number of ways to place 3 rooks anywhere in } B)\\ &= (n-3)! \cdot (\text{Coefficient of } x^3 \text{ in } R_B(x))\\ &= (n-3)! \cdot r_3\end{aligned}$$

Keep in mind that the $(n-3)!$ term comes from the ways to freely place the remaining rooks. This same method could be used for any number of rook placements. If we let r_k be the number of ways to place k rooks in B, we can see that (1) above, which represents the coefficient of x^n in $R_{\overline{B}}(x)$, can clearly be written

$$n! - (n-1)!r_1 + (n-2)!r_2 - ... + (-1)^n 0! r_n.$$

Thus the theorem is proved by using the Inclusion-Exclusion Principle. □

Notice that Theorem 4 only says B is part of an $n \times n$ board. We mentioned that this property will also hold if we think of an $n \times n$ board as an $m \times n$ board with $n - m$ rows blacked out. However, while this is helpful in understanding the theorem, it is not necessary to black out these squares as the Inclusion-Exclusion Principle is defined for a specific sub-board B and will still exclude these new squares as possible rook placements. Because of the way we defined N, we are only counting rooks placed inside B. It does not matter what the rest of the $n - m$ rows look like as they are not a part of B. Furthermore, because our theorem only gives the number of ways to place n rooks, unless the board is $n \times n$ then the number of ways to place n rooks will be 0. We cannot have $m > n$ because then it could not be a sub-board, and if $m < n$ then we cannot fit n rooks onto the board at all. While it will work for rectangular boards, it is neither helpful nor interesting. This theorem allows for some interesting results.

Example 4. *How many derangements are there of size n?*

A derangement is just a permutation where no element is permuted into its same position. For example, $1 \to 1$ is not permitted. We can convert this problem into one involving rook polynomials. The board on the left in Figure 2.8 represents permutations where an element is not allowed to be permuted to itself.

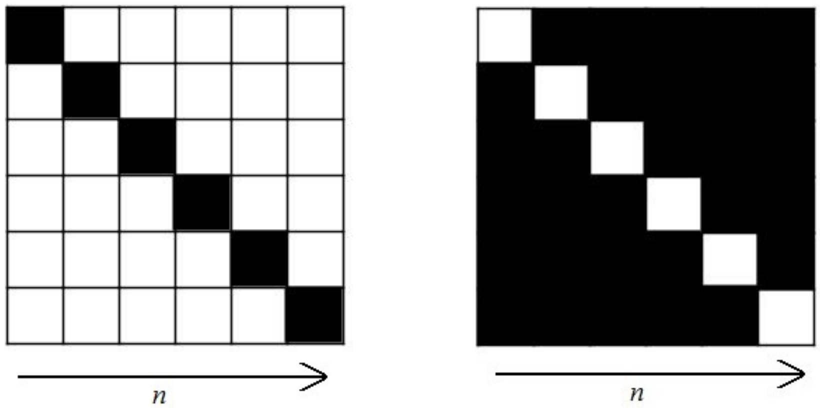

Figure 2.8: The board for derangement permutations and its complement.

In its current form, it would take a very long time to calculate this board's rook polynomial so that we could find the coefficient of x^n and answer our question. However, given our handy Theorem 4, we can find the coefficient r_n without having to find the whole rook polynomial. First look at the complement of this board as shown on the right in Figure 2.8.

Let's call the left board D and the right board $\overline{D}$. Looking at $\overline{D}$ we can clearly see that it is separable. The rook polynomial for such a board is $R_{\overline{D}}(x) = (1 + x)^n$. Now we can use the Binomial Theorem,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

to show that the coefficient of x^k in the rook polynomial $R_{\overline{D}}(x)$ will be $\binom{n}{k}$ if we set $a = x$ and $b = 1$. Using this result and Theorem 4 we can see that the number of derangements of length n is exactly

$$n! \cdot \binom{n}{0} - (n-1)! \cdot \binom{n}{1} + (n-2)! \cdot \binom{n}{2} - \cdots + (-1)^n \cdot 0! \cdot \binom{n}{n}.$$

If we use the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

we get an interesting result. Folding it into the use of Theorem 4 we see that

$$\begin{aligned}
&n!\binom{n}{0} - (n-1)! \cdot \binom{n}{1} + (n-2)! \cdot \binom{n}{2} - \cdots + (-1)^n \cdot 0! \cdot \binom{n}{n} \\
&= \frac{n!}{0!} - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!} \\
&= n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}\right) \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!},
\end{aligned}$$

which is a common way to express the number of size n derangements.

n	0	1	2	3	4	5	6	7	8	9
D_n	1	0	1	2	9	44	265	1854	14833	133496

Table 2.1: Shows the number of derangements D_n of sets with n elements.

Theorem 4 is very helpful when we are concerned with only the maximal number of rooks we can place. However, what if we want to know about how many ways we can place less than the maximum number of rooks on B? Our theorem would not be very helpful. It turns out there is a formula to find the number of ways to place k rooks on B if we know the rook polynomial of $\overline{B}$.

The following theorem and its proof have been drawn from Feryal Alayont [1].

Theorem 5. *Let B be a sub-board of a full $m \times n$ board and let $\overline{B}$ be its complement. Let $R_B(x) = \sum_{i=0}^{k} r_i(B)x^k$ be the rook polynomial of B. The number of ways to place k non-attacking rooks on $\overline{B}$ is*

$$r_k(\overline{B}) = \sum_{i=0}^{k} (-1)^i \cdot \binom{m-i}{k-i} \cdot \binom{n-i}{k-i} \cdot (k-i)! \cdot r_i(B).$$

We set $r_i(B) = 0$ whenever i is greater than the degree of the rook polynomial of B.

Proof. This proof uses a similar argument as our other complement theorem, but will have some key differences. In order to find the number of ways to place k non-attacking rooks on $\overline{B}$, we will look at all the placements of k non-attacking rooks on the full $m \times n$ board and then remove those placements where one or more rooks are placed on B. We can accomplish this by using the Inclusion-Exclusion Principle. In order to simplify this calculation, we will temporarily number the rooks. This means that now we are looking for $k! \cdot r_k(\overline{B})$ with the additional $k!$ term multiplied to account for this numbering of rooks. We know from Theorem 1 that the number of ways to place k rooks on the unrestricted $m \times n$ board is

$$\binom{m}{k} \cdot \binom{n}{k} \cdot (k!)^2.$$

We squared the $k!$ term to account for the numbering of the rooks again. Let A_i denote the set of placements of the rooks where the ith rook is on B. We need to remove these placements from the set of all placements. We know that there are $r_1(B)$ ways to place the ith rook on B and

$$\binom{m-1}{k-1} \cdot \binom{n-1}{k-1} \cdot ((k-1)!)^2$$

ways to place the other rooks in the remaining rows and columns unrestricted. Now we know that there are

$$r_1(B) \cdot \binom{m-1}{k-1} \cdot \binom{n-1}{k-1} \cdot ((k-1)!)^2$$

elements in every A_i and there are exactly k values for i which gives us k different A_i's. Similarly, there are

$$r_2(B) \cdot 2! \cdot \binom{m-2}{k-2} \cdot \binom{n-2}{k-2} \cdot ((k-2)!)^2$$

elements in $A_i \cap A_j$ for any $i \neq j$. We also know that there are $\binom{k}{2}$ double intersections with 2! ways to place the numbered rooks. Following this logic, and using the Inclusion-Exclusion principle, where we remove single intersections, add double intersections, remove triple intersections, and so on, we get that the number of ways to place these k numbered rooks on $\overline{B}$ is

$$\sum_{i=0}^{k} (-1)^i \cdot \binom{k}{i} \cdot i! \cdot \binom{m-i}{k-i} \cdot \binom{n-i}{k-i} \cdot ((k-i)!)^2 \cdot r_i(B).$$

Now we can divide this by $k!$ in order to get rid of the numbering we put on the rooks. This leads us to

$$\begin{aligned} r_k(\overline{B}) &= \sum_{i=0}^{k}(-1)^i \cdot \binom{k}{i} \cdot i! \cdot \binom{m-i}{k-i} \cdot \binom{n-i}{k-i} \cdot ((k-i)!)^2 \cdot r_i(B) \cdot \frac{1}{k!} \\ &= \sum_{i=0}^{k}(-1)^i \cdot \frac{k!}{i!(k-i)!} \cdot i! \cdot \binom{m-i}{k-i} \cdot \binom{n-i}{k-i} \cdot ((k-i)!)^2 \cdot r_i(B) \cdot \frac{1}{k!} \\ &= \sum_{i=0}^{k}(-1)^i \cdot \binom{m-i}{k-i} \cdot \binom{n-i}{k-i} \cdot (k-i)! \cdot r_i(B). \end{aligned}$$

Now that we have removed our temporary numbering of rooks, our proof is concluded. □

One of the significant differences between this theorem and Theorem 4 is that we are now no longer just dealing with square boards. This will work for any board and its complement for any coefficient we want. As we saw when working with derangements, there are some interesting problems that can be solved with rook board complements. We can do a small test of this theorem on one of the coefficients from our boards in Figure 2.7. Recall that the two rook polynomials are

$$R_B(x) = 1 + 14x + 64x^2 + 112x^3 + 68x^4 + 9x^5$$

$$R_{\overline{B}}(x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 3x^5.$$

We will use Theorem 5 to find $r_2(\overline{B})$. Because we are dealing with an $n \times n$ board, we can just change m to n in our formula. In this example, $n = 5$ and we get

$$
\begin{aligned}
r_2(\overline{B}) &= \sum_{i=0}^{2} (-1)^i \cdot \binom{5-i}{2-i} \cdot \binom{5-i}{2-i} \cdot (2-i)! \cdot r_i(B) \\
&= \sum_{i=0}^{2} (-1)^i \cdot \binom{5-i}{2-i}^2 \cdot (2-i)! \cdot r_i(B) \\
&= (-1)^0 \cdot \binom{5-0}{2-0}^2 \cdot (2-0)! \cdot r_0(B) \\
&+ (-1)^1 \cdot \binom{5-1}{2-1}^2 \cdot (2-1)! \cdot r_1(B) \\
&+ (-1)^2 \cdot \binom{5-2}{2-2}^2 \cdot (2-2)! \cdot r_2(B) \\
&= (10^2 \cdot 2 \cdot 1) - (4^2 \cdot 1 \cdot 14) + (1^2 \cdot 1 \cdot 64) \\
&= 200 - 224 + 64 \\
&= 40 = r_2(\overline{B}).
\end{aligned}
$$

We could do this for each coefficient, but that would be a very tedious calculation.

Chapter 3

The Problem of the Bishops

3.1 The Maximal Number of Bishops and Bishop Placements on Square Boards

What if instead of placing rooks on a board, we placed bishops? How many non-attacking bishops can we place on an $n \times n$ board and in how many different ways are we able to place them?

For those that are unfamiliar with chess, a bishop is much like a rook, but can only move along diagonals. This is much different than before when dealing with rooks as now we cannot switch rows and columns. In the rook problem, switching a row and column yielded the same polynomial, but when dealing with bishops this will not always be the case. If we think about a standard chess board, we have black and white alternating squares. A bishop placed on a white square can never attack a bishop on a black square. If we permute rows or columns it would be easy to have two bishops that should never be able to attack each other appearing in the same diagonal. In order to handle this issue, we convert the problem with bishops into one with rooks. It is possible to restrict placement of bishops to change the board, but we will not discuss that here. Before we begin discussing placing bishops onto boards, we need a few definitions.

Definition 5. *Let B be a board of size $m \times n$. The* ***bishop polynomial*** *of B is defined as*

$$B_B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_s x^s,$$

where b_k is the number of ways to place k non-attacking bishops on B. When B is $n \times n$ we will denote the bishop polynomial as $B_n(x)$.

It should be noted that the reason the highest term in this polynomial has degree s (as opposed to m or n) is because more bishops can be placed on the board than there are rows or columns. As the board dimensions change, s will change. Later we will find a formula for s when the board is $m \times n$ and unrestricted.

Now let us try and determine a way to transform the problem into one with rooks. In order to separate the boards, start at the top left corner of the board. Then moving from left to right alternate coloring the squares with black and white. Now move to the next row and color each square opposite the color of the square above it and repeat this second step until the whole board is colored. Now we have a board that looks much like a chess board. Keep in mind that we are not blacking out these squares as we did in the previous chapter, we are simply labeling the sub-boards with colors and bishops are still able to be placed in any square. Because of the way bishops can attack on the board, a bishop placed on a black square will never be able to attack a bishop placed on a white square and vice versa. We can separate these into two independent boards. Consider the 4×4 board in Figure 3.1. A careful examination will see that any bishop placed in the top board will correspond to a rook placed on the corresponding letter of the bottom board. We have shown that bishop boards can be split into two independent rook boards. Because these are two rook boards, we can find their rook polynomials and multiply them together to get the bishop polynomial as shown in Theorem 6.

Theorem 6. *If B is an $m \times n$ board for placing bishops, then*

$$B_B(x) = R_w(x) \cdot R_b(x),$$

where $R_w(x)$ is the rook polynomial of the white colored sub-board and $R_b(x)$ is the rook polynomial of the black colored sub-board.

Proof. Consider coloring B in the alternating fashion described earlier. The result will be a chessboard style coloring with black and white squares. Note again that no bishop placed on a black square can attack a bishop on a white square and vice versa. Also note that these sub-boards are equivalent to rook boards. By turning the boards by 45 degrees, we can see that a bishop can only move up, down, left, and right just like a rook. This lends itself to a fairly simple translation to a rook board. Because these two sub-boards are independent with respect to bishop placement, in order to compute the bishop polynomial we simply need to multiply the resulting rook polynomials from the independent sub-boards. □

To begin, we are interested in the leading coefficient of the output polynomial $B_n(x)$. We will examine the rest of the polynomial later on in Section 3.2. The two rook boards in Figure 3.1 are identical, so we can just square the rook polynomial of one board to find the bishop polynomial as shown in Theorem 6. The bishop polynomial is

$$B_4(x) = (1 + 8x + 14x^2 + 4x^3)^2 = 1 + 16x + 92x^2 + 232x^3 + 260x^4 + 112x^5 + 16x^6.$$

This means that we can place a maximum of 6 bishops in 16 different ways. Notice that the maximal number of ways we can place the bishops in this example is 2^4. This turns out to be no coincidence as we will show later.

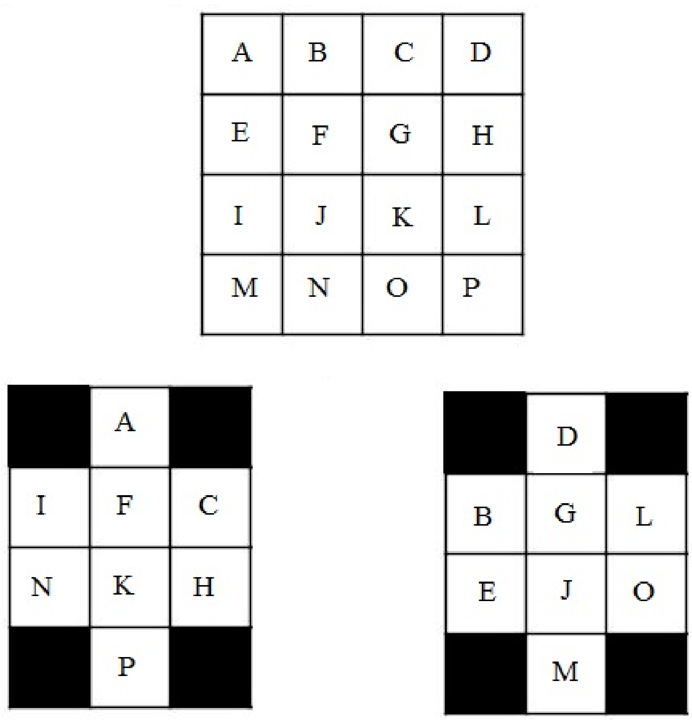

Figure 3.1: Separation of a 4×4 bishop board into two rook boards. The sub-boards have been rotated to demonstrate their identical construction.

3.1.1 Square Bishop Boards when n is Even

Now let us look at all $n \times n$ boards when n is even. Consider coloring the board starting with white in the top left corner. If you look at an even $n \times n$ board reflected about the center, we have an identical but opposite coloring. This means that the two colorings, black and white, will produce identical rook sub-boards. When n is even, the two sub-boards will have identical rook polynomials. Let us now try and determine what these boards will look like in general.

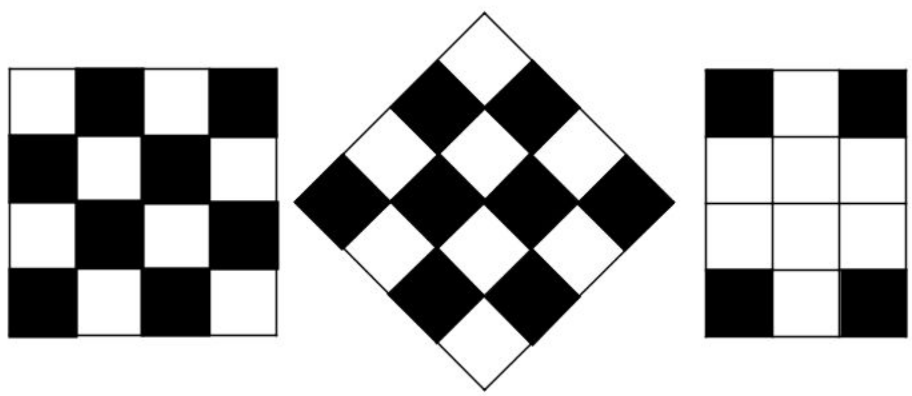

Figure 3.2: Rotating the original bishop board to find a rook sub-board.

Look at Figure 3.2. If we rotate the bishop board on the left to see the board in the middle, we can see that now bishops move just like rooks. Now look at just the white squares in the middle board. We can take just these white squares and form them into the rook board on the right in Figure 3.2. Similarly if you look at the black squares, we can see we would get an identical rook board just rotated onto its side. While this example was just the 4×4 case, think about how this board will look for any even n. We will have a single square in the top row, followed by 3 in the second row, where the top row can only attack the middle of these three. Then in the third row we will get 5 squares where they begin to look like a pyramid. When we reach the middle, we will have two rows with $n-1$ open squares. After this repeated row we follow the same pattern as before but in reverse until we reach the last row which will only have one square again. This leads us to another definition.

Definition 6. *Let n be even. A* ***mixed diamond board*** *is a board with n rows and* $n-1$ *columns where the open squares in each row follow the pattern* $1, 3, 5, ..., n-1, n-1, ..., 5, 3, 1$ *and form a diamond shape. We will denote the rook polynomial of these boards by* $M_n(x)$.

Now we can examine the rook polynomials in general for these mixed diamond boards. Because the board is always $n \times n-1$, we know that the upper bound on the maximum possible number of rooks we can place is $n-1$. We must show we can place this many rooks on these boards, or we must show that in the rook polynomial of the mixed diamond board that the coefficient of x^{n-1} is nonzero. Consider the following placement of rooks: place a rook in row $\frac{n}{2}$ and column 1. Then place rooks diagonally moving upward 1 and to the right 1 until you reach row 1. You should now have $\frac{n}{2}$ non-attacking rooks on the board. Now begin placing rooks in row $\frac{n}{2}+1$ and column $n-1$. Place your remaining rooks along the diagonal moving downward 1 and to the left 1 until you reach row $n-1$. Now you should have placed a total of $n-1$ rooks on the board. This means that the coefficient of x^{n-1} is nonzero. The final result is shown in Figure 3.3.

Now that we have shown that the degree of $M_n(x)$ is $n-1$, when we square $M_n(x)$, the degree of the bishop polynomial will be $2n-2$. This means that

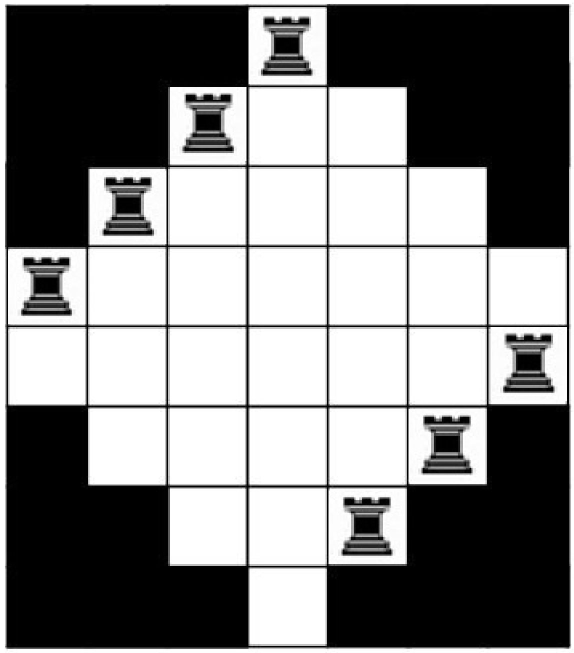

Figure 3.3: Mixed diamond board showing a maximal placement of rooks.

when n is even, we can place at most $2n - 2$ bishops on an $n \times n$ sized board. But in how many different ways can we place them?

Lemma 1. *If m_{n-1} is the coefficient of x^{n-1} in $M_n(x)$, then $m_{n-1} = 2^{n/2}$.*

Proof. We will prove this lemma using induction. Looking at the mixed diamond boards, in order to place $n - 1$ rooks, we must have one rook in each column. Consider a mixed diamond of size 2×1. This corresponds to a bishop board of size 2×2. We know that there are only $2 = 2^{2/2}$ ways to place a rook on this board. This is our base case. We assume for a $k - 2 \times k - 3$ mixed diamond board that our lemma is true and the leading coefficient is $2^{(k-2)/2}$. We want to show our lemma holds for a $k \times k - 1$ mixed diamond. In other words, we assume that the leading coefficient in $M_{k-2}(x)$ is $2^{k-2/2}$ to show that the leading term in $M_k(x)$ is $2^{k/2}$. Notice that because we are only dealing with even k values, for our induction hypothesis we need to decrease k by 2 to get the next lower case. Now consider a $k \times k-1$ mixed diamond board. If we look at the first column, there will be 2 open squares. There are 2 options to place a rook in the first column. When we choose one of these squares, it forces our choice in the last column. We now see that after placing these rooks, we can no longer place rooks in these middle two rows. If we remove these rows, we have a $k-2 \times k-3$ mixed diamond board. We can multiply the number of ways to place rooks in this smaller mixed diamond, which we know from our induction hypothesis, by the 2 options to place rooks in the first and last columns to see that

$$m_k = 2 \cdot 2^{(k-2)/2} = 2^{k/2}.$$

By induction, the number of ways to place $n-1$ rooks on the mixed diamond board of size n is $2^{n/2}$. This means the leading term in the rook polynomial $M_n(x)$ will be $2^{n/2}x^{n-1}$. □

When we square the rook polynomial, the leading term in the bishop polynomial $B_n(x)$ will be $2^n \cdot x^{2n-2}$. When n is even, we can place at most $2n-2$ bishops in 2^n different ways. Now what happens when n is odd?

3.1.2 Square Bishop Boards when n is Odd

Let us start to think about this with an example. If we look at a 5×5 board such as the one in Figure 3.4, we can see that the two rook sub-boards are not identical. Again we are still interested in only the leading term of their respective rook polynomials.

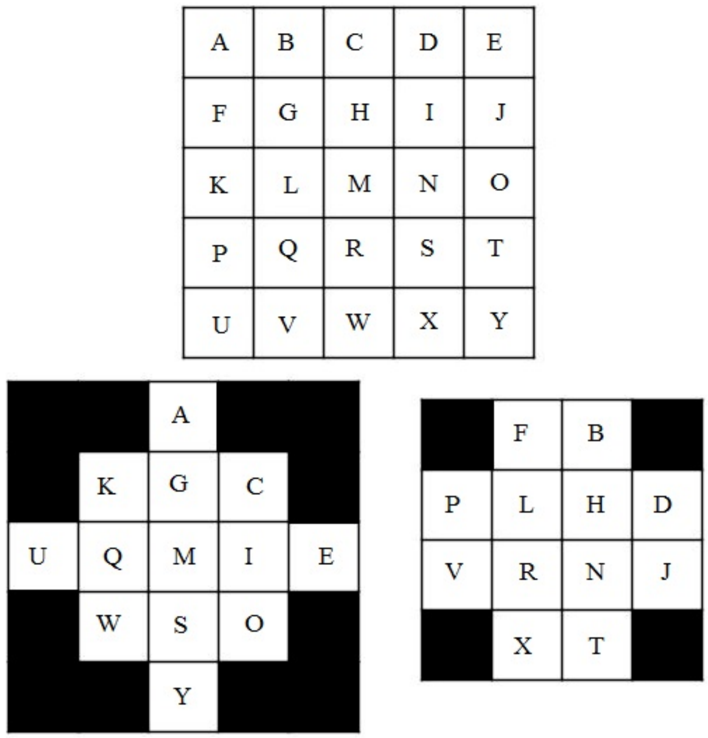

Figure 3.4: Odd sized bishop boards do not separate into identical rook sub-boards.

In general let us find the forms of these two sub-boards as we did for even sized bishop boards. Because the number of rows and columns is odd, all the corners will be of one color. We can see that one board will be larger than the other. Coloring the top left corner white, let us look at the white sub-board. As we did in Figure 3.2, turn the board over into a diamond and separate into two boards again. We get a similar pattern to the mixed diamond. The key difference is that we have a middle row with n open squares and there is only one such row. In row 1 we have 1 free square in the middle column. Then in row 2 we have 3 open squares in the middle of the row. This continues until row $\frac{n+1}{2}$, where we have n free squares. Then continuing on we shrink back in a reversed manner until there is only 1 square in row n. We will call the resulting board an odd diamond.

Definition 7. *Let n be odd. An* ***odd diamond board*** *is an $n \times n$ rook board where the open squares in each row follow the pattern $1, 3, 5, ..., n-2, n, n-2, ..., 5, 3, 1$ and form a diamond shape. We will denote the rook polynomials of these boards by $O_n(x)$.*

With the top left square colored white, let us now look at the black sub-board. Using the same trick again of rotating the board and looking at the resulting rook board, we can see that our first row will have 2 free squares. Our second will have 4, and so on until we reach the two middle rows each with $n-1$ free squares. Then we shrink back down to 2 free squares in row $n-1$. We will call the form of this rook board an even diamond.

Definition 8. *Let n be odd. An* ***even diamond board*** *is an $n-1 \times n-1$ rook board where the open squares in each row follow the pattern $2, 4, 6, ..., n-1, n-1, ..., 6, 4, 2$ and form a diamond shape. We will denote the rook polynomial for these boards by $E_n(x)$.*

Let us first examine the even diamond boards. Because the board is $n-1 \times n-1$, we need to determine if the degree of the rook polynomial is $n-1$. We know that $n-1$ is the upper bound on the number of rooks we can place. We now need to determine that in the rook polynomial the coefficient of x^{n-1} is nonzero. Using a similar method for how we described placing rooks on mixed diamond boards, and shown in Figure 3.4, we can only place a maximum of $n-1$ rooks. Thus the coefficient of x^{n-1} is nonzero. One way to place rooks on an even diamond is shown in Figure 3.5.

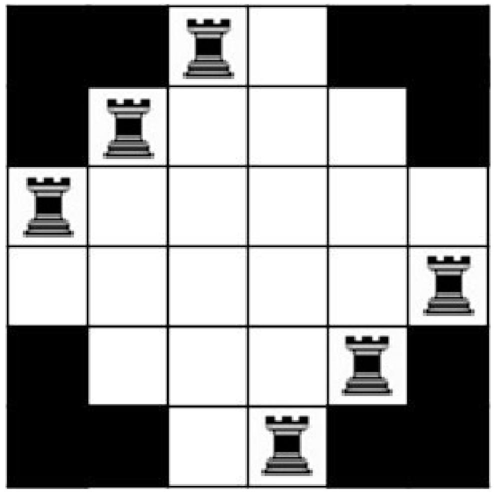

Figure 3.5: Even Diamond board showing a maximum placement of rooks.

Now let us look at the number of ways to place these $n-1$ rooks.

Lemma 2. *If e_{n-1} is the coefficient of x^{n-1} in $E_n(x)$, then $e_{n-1} = 2^{(n-1)/2}$.*

Proof. This proof is almost identical to that of mixed diamonds. Because these boards are $n-1 \times n-1$, we need one rook in each column. Consider that when $n = 3$ we have a 2×2 even diamond board. Clearly there are $2 = 2^{(3-1)/2}$ ways to place 2 rooks on this board. If we assume that our lemma holds for a $k-3 \times k-3$ board, we can show it is true for $k-1 \times k-1$ board. We assume for a $k-3 \times k-3$ board the leading coefficient of the rook polynomial is $2^{(k-3)/2}$ Consider an even diamond board of size $k-1 \times k-1$ while placing a rook in the first column. Just as before we have 2 options and when we choose one, it forces our choice in the last column. Now when we have rooks in the first and last columns we cannot place any rooks in the middle rows. If we remove these rows, we see that the board will collapse down into an even diamond board with dimensions $k-3 \times k-3$. From our induction hypothesis, we know that the number of ways to place rooks on this smaller board is $2^{(k-3)/2}$. We multiply it by 2 to see that for a board of size k

$$e_k = 2 \cdot 2^{(k-3)/2} = 2^{(k-1)/2}.$$

Thus the leading term in the rook polynomial $E_n(x)$ will be $2^{(n-1)/2}x^{n-1}$. □

Odd diamonds are slightly more confusing to analyze. We have that for an $n \times n$ sized bishop board, the odd diamond sub-board is also size $n \times n$. However in this case, we will not be able to place n rooks on this board. If you think about trying to place a rook in each column, once we place a rook in column 1, there is nowhere to place a rook in column n. Because the board is $n \times n$ and we cannot fit a rook in each column, the coefficient in the rook polynomial of x^n will always be 0. Now let us see if the coefficient of x^{n-1} is also 0. Consider the following placement of rooks. Place a rook in the only square in column 1. Then place along the diagonal moving up one and right one square until you reach the first row. This means we have placed $\dfrac{n+1}{2}$ rooks. Then continuing, place a rook in row $n-1$ and column $\frac{n+1}{2}$. Continue moving up in the same manner as before and terminating at column $n-1$. This will correspond to placing $n-1$ rooks on the board. Thus the coefficient of x^{n-1} is nonzero.

Let us now find the number of ways to place these $n-1$ rooks on the board.

Lemma 3. *Let $n > 1$. If o_{n-1} is the coefficient of x^{n-1} in $O_n(x)$, then $o_{n-1} = 2^{(n+1)/2}$.*

Before getting into the proof, we must note that in the case of $n = 1$, we can place 1 rook which is not equal to $2(1) - 2 = 0$. The reason it doesn't work for

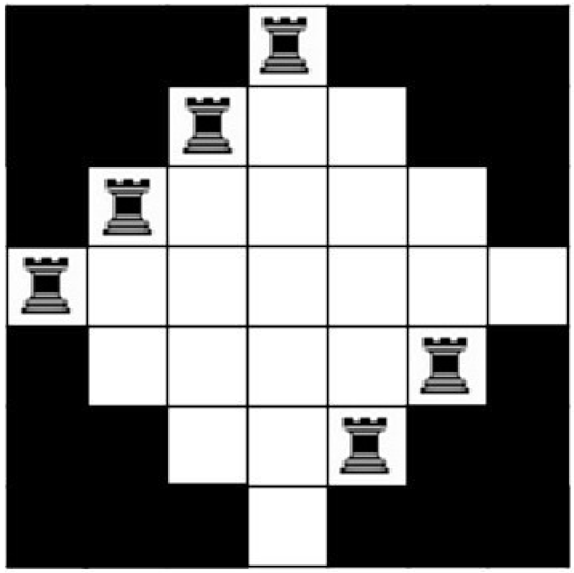

Figure 3.6: Rooks on an odd diamond board showing the coefficient of x^{n-1} is nonzero.

$n = 1$ is because this board will not separate into two boards. However, because a single square bishop board is rather uninteresting, we can move on to larger boards.

Proof of Lemma 3. Consider an odd diamond of size 3×3. In order to fit two rooks on this board, we must have the rooks be on the edge. If we put one in the middle square, then we could not place a second. Now there can be only one in the middle row and one in the middle column as well. It is a simple matter of counting to see that there are $4 = 2^{(3+1)/2}$ ways to place these rooks. Let this be our base case. For our induction argument to work it is important to keep in mind that to have a smaller case, we need to decrease k by 2 because we are only dealing with odd dimensions. We can then assume that for a board of size $k - 2 \times k - 2$, the most ways we can place rooks is $2^{(k-1)/2}$. We want to show that our lemma holds for a board of size $k \times k$. Consider an odd diamond of size $k \times k$ with a placement of $k-1$ rooks. There must be one in the middle row and it must be on one of the edge columns as this will restrict the fewest number of remaining squares. There are only 2 ways to place a rook on the edge of the middle row. Now we can remove the middle row as we can not place any more rooks there. This leaves us with a mixed diamond of size $k-1 \times k-2$. We know that the number of ways to place $k-2$ (the maximum number) rooks on this board is $2^{(k-1)/2}$ from Lemma 1. When we multiply this by the 2 ways we mentioned to place a rook in the middle row, we see that

$$o_k = 2 \cdot 2^{(k-1)/2} = 2^{(k+1)/2}.$$

By this induction argument, our lemma is proved. □

This means that the leading term for the rook polynomial of an odd diamond is $2^{(n+1)/2} \cdot x^{n-1}$. Then the leading term in the bishop polynomial is

$$e_{n-1}x^{n-1} \cdot o_{n-1}x^{n-1} = 2^{\frac{n-1}{2}}x^{n-1} \cdot 2^{\frac{n+1}{2}}x^{n-1} = 2^n x^{2n-2}.$$

This is exactly the same as what we found for even sized boards. The leading term of the bishop polynomial is not dependent on the parity of the size of the board. We can state this observation as a theorem.

Theorem 7. *Let $n > 1$. If B_n is an $n \times n$ board for placing bishops, then the maximum number of non-attacking bishops that can be placed on B_n is $2n-2$ and they can be placed in 2^n different ways.*

The proof for Theorem 7 directly follows from the results of Lemmas 1, 2, and 3 using Thoerem 6 to find the leading term of the bishop polynomial $B_n(x)$. What if we wanted to know how many ways to place, m bishops on B_n? Theorem 7 and the preceding Lemmas would not be very useful in answering this question. Let us now investigate a way to find the entire polynomial $B_n(x)$ in general for any $n > 1$.

3.2 The General Bishop Polynomial for Square Boards

Before discussing a general formula for the Bishop Polynomial, there are a few other topics we need to discuss, the first of which is the Triangle Board as shown in Figure 3.7

Definition 9. *A **triangle board** is an $n \times n$ board for rook placement where the diagonal and every square beneath it are all open, while every square above the diagonal is restricted for rook placement. We will denote the rook polynomial for these boards by $T_n(x)$.*

There is a general formula for $T_n(x)$, but before we can discuss what that is, we need to introduce another topic: the Stirling numbers of the second kind. The definition of these numbers is taken from Victor Bryant [3].

Definition 10. ***The Stirling number of the second kind**, denoted $S(n,k)$, is equal to the number of different ways of partitioning a set of n elements into k non-empty subsets of any size.*

These numbers have many interesting recurrence relationships and

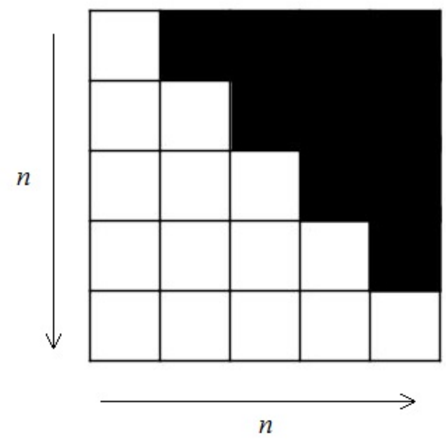

Figure 3.7: The Triangle Board T_n.

		k							
		0	1	2	3	4	5	6	7
n	0	1							
	1	0	1						
	2	0	1	1					
	3	0	1	3	1				
	4	0	1	7	6	1			
	5	0	1	15	25	10	1		
	6	0	1	31	90	65	15	1	
	7	0	1	63	301	350	140	21	1

Table 3.1: Table of values for $S(n,k)$.

various properties but we only need one of them for our purposes. The following theorem and proof are taken from van Lint and Wilson [12].

Theorem 8. *The Stirling numbers of the second kind satisfy the following recurrence relation*

$$S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1).$$

Proof. A partition of the set $\{1, 2, 3, ..., n-1\}$ can be made into a partition of

$\{1, 2, ..., n\}$ by adjoining n to one of the k blocks or by increasing the numer of blocks by one making $\{n\}$ a block. We can see that there are k ways to adjoin n to the different blocks. This is where we get $k \cdot S(n-1, k)$ as we have $n-1$ elements and need to make k blocks with k ways to adjoin n to a block. Then if $\{n\}$ is its own block, then we only need $k-1$ other blocks. This is where we get $S(n-1, k-1)$. Adding these two options together, we have proven the theorem. □

The reason we need to understand these Stirling numbers is that they are the basis for the formula for $T_n(x)$ [3]. The proof of Theorem 9 will draw from the proof described by Feryal Alayont [1].

Theorem 9. *For any $n > 0$,*

$$T_n(x) = \sum_{j=0}^{n} S(n+1, n+1-j)x^j.$$

Proof. Essentially, we are trying to prove the statement: "The number of ways to place j rooks on a triangle board of size n is equal to $S(n+1, n+1-j)$ where $0 \leq j \leq n$." We will prove this using induction on n. Our base case will be a triangle board of size 1 which has a rook polynomial of $1 + x$. This fits into our theorem as $S(2, 2) = 1$ and $S(2, 1) = 1$.

We can now assume that the number of ways to place k rooks on a size n triangle board is equal to $S(n+1, n+1-k)$ where $0 \leq k \leq n$. What we want to show that the number of ways to place k rooks on an $(n+1) \times (n+1)$ triangle board where $0 \leq k \leq n+1$ is equal to $S(n+2, n+2-k)$. When $k = n+1$ only one rook placement where all rooks are on the diagonal will fit all k rooks. This corresponds to $S(n+2, n+2-(n+1)) = S(n+2, 1) = 1$. When $k = 0$, there is only one way to place k rooks on the board which also corresponds to $S(n+2, n+2) = 1$. When we have a size $(n+1) \times (n+1)$ triangle board, our formula agrees with $k = n+1$ and $k = 0$.

When $0 < k < n+1$ we will have two cases. The first case is when we place our k rooks in the top n rows. In this case, we will have a size n triangle board. From our induction hypothesis, we know that the number of ways to do this is $S(n+1, n+1-k)$. Our second case is when there is one rook in the bottom row. This means that we need to place $k-1$ rooks on the size n triangle board above. There are $S(n+1, n+1-(k-1)) = S(n+1, n+2-k)$ ways to do this based on our induction hypothesis. This leaves us with $(n+1)-(k-1) = n+2-k$ open squares in the bottom row in which place our last rook. When the last row is used, we have $(n+2-k)S(n+1, n+2-k)$ ways to place k rooks on the board. Adding these two cases together and using Theorem 8 we see that the total number of ways to place k rooks on the board is

$$(n+2-k) \cdot S(n+1, n+2-k) + S(n+1, n+1-k) = S(n+2, n+2-k).$$

Therefore by induction, the number of ways to place k rooks on a size n triangle board is $S(n+1, n+1-k)$. This is simply the coefficient of x^k in $T_n(x)$.

□

Now that we have discussed the basis material, we can finally introduce our general formula for $B_n(x)$.

Theorem 10. *For any integer $n > 1$,*

$$B_n(x) = \sum_{q=0}^{2n-2} \sum_{s=0}^{q} \left(\sum_{l=0}^{s} \binom{\lceil n/2 \rceil}{l} S(n-l, n-s) \right) \cdot \left(\sum_{h=0}^{q-s} \binom{\lfloor n/2 \rfloor}{h} S(n-h, n-(q-s)) \right) x^q.$$

The proof of Theorem 10 must come later as we will need a few lemmas before we can get started. As for now, consider the formula in the light of our findings from the previous section on maximal bishop placement. We determined that the parity did not change the end result for maximal rooks. We combine these two cases of even and odd into one case using a ceiling and floor function because of what we will find in the case of odd sized boards. We will not state this as a piecewise defined function here because the formula is much cleaner this way. Much like we did in the previous sections, let us split our focus in these matters to when n is either even or odd. This will give a better understanding of how to arrive at the above formula.

3.2.1 The Polynomial $B_n(x)$ when n is Even

As we learned from Theorem 6, $B_n(x)$ is equal to the product of rook polynomials of two independent rook sub-boards. Also recall that when dealing with an even integer n, the two resulting sub-boards are identical mixed diamond boards. Examining these boards more carefully, notice that we can rearrange the rows and columns to produce a board that resembles a triangle board. The rook polynomial will not change when we do this. In the first two rows we will have 1 open square each, then the next two rows each have 3 open squares and so on until we reach the bottom two rows that will each have $n-1$ open squares. If we add an extra column of restricted squares on the right, we have an $n \times n$ rook board that closely resembles a triangle board. An example is shown in Figure 3.8. Our goal is to use the formula we

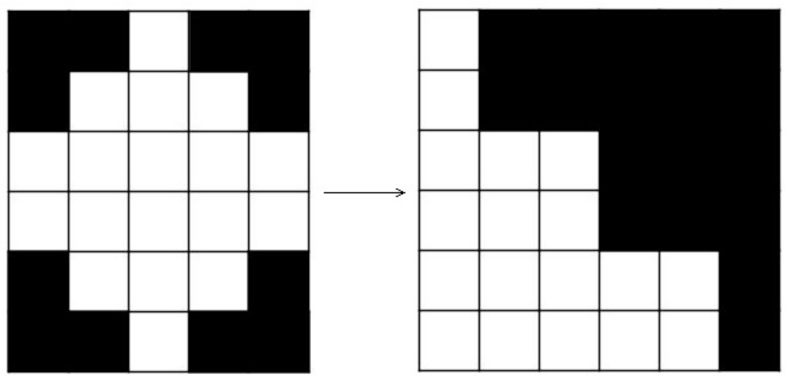

Figure 3.8: Rows and columns of a mixed diamond board rearranged.

already have for triangle boards to find a formula for mixed diamond boards. Notice that the only difference between the two boards is the diagonal. In the triangle board, the diagonal is all open, while in the mixed diamond, alternating squares are blacked out. It turns out that exactly $\frac{n}{2}$ squares will be blacked out. Let us state the formula for $M_n(x)$ as a lemma and then show how to find it in the proof.

Lemma 4. *If n is even, then*

$$M_n(x) = \sum_{i=0}^{n/2} \binom{n/2}{i} T_{n-1-i}(x) \cdot x^i.$$

Proof. We will prove this by using a recursive relationship featuring triangle boards. First, let M_n be our $n \times n$ mixed diamond board with rearranged columns and rows in the style of Figure 3.8. Consider the rook polynomial of M_n if we place no rooks on the diagonal. This means we restrict the $\frac{n}{2}$ open squares on the diagonal and only place rooks below that. Notice that this board is a triangle board of size $n-1$ and the rook polynomial is exactly $T_{n-1}(x)$.

Now consider placing just one rook in any of the open squares on the diagonal. Because we have $\frac{n}{2}$ open squares, there are $\binom{n/2}{1}$ ways to choose an open square. Now look at what happens to our board when we place one rook on the diagonal and restrict the remaining open squares for the moment. Figure 3.9 shows what the resulting board will look like where the gray squares are usually open, but because we are choosing just one of them on which to place a rook, they are restricted temporarily. By placing a rook on the

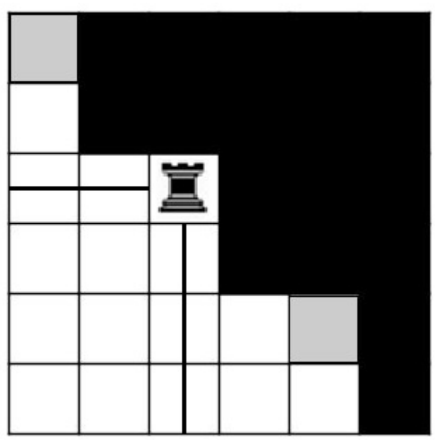

Figure 3.9: Mixed diamond board with one rook on the diagonal and other diagonal squares restricted.

diagonal, we cannot place any more rooks in that row or column. Also notice that the remaining open squares form another smaller triangle board. It does not matter which open square we choose to place our rook in, the rook we place will always prevent us from placing more rooks in one row and one column. We can add to our polynomial $\binom{n/2}{1}T_{n-2}(x) \cdot x$. We multiply by x because we are placing just one rook on the board so we would have to raise the power of every term in T_{n-2} by 1.

We can see that we get a similar result with 2 rooks. When we collapse the open squares again, we get a smaller triangle board the more rooks we place. The same process for 2 rooks will yield $\binom{n/2}{2}T_{n-3}(x) \cdot x^2$. We can continue all the way until we have rooks placed on every open square on the diagonal. Our final term would be $\binom{n/2}{n/2}T_{n/2-1}(x) \cdot x^{n/2}$. Combining these terms we get

$$\begin{aligned} M_n(x) &= \binom{n/2}{0}T_{n-1}(x) + \binom{n/2}{1}T_{n-2}(x) \cdot x + \binom{n/2}{2}T_{n-3}(x) \cdot x^2 + \cdots + \binom{n/2}{n/2}T_{n/2-1}(x) \cdot x^{n/2} \\ &= \sum_{i=0}^{n/2} \binom{n/2}{i} T_{n-1-i}(x) \cdot x^i. \end{aligned}$$

□

Now that we have this formula, we can substitute in stirling numbers for the triangle board polynomial using Theorem 9 to find the bishop polynomial for mixed diamonds in terms of stirling numbers. Substituting, we get

$$\begin{aligned} M_n(x) &= \sum_{i=0}^{n/2} \binom{n/2}{i} T_{n-1-i}(x) \cdot x^i. \\ &= \sum_{i=0}^{n/2} \binom{n/2}{i} \left[\sum_{j=0}^{n-1-i} S(n-i, n-i-j)x^j \right] x^i \\ &= \sum_{i=0}^{n/2} \sum_{j=0}^{n-1-i} \binom{n/2}{i} S(n-i, n-(i+j))x^{i+j}. \end{aligned}$$

Now this is a slightly difficult result to work with. What we want is just a formula for the coefficient of x^k in $M_n(x)$. This leads us to our next lemma which we will use to prove Theorem 10.

Lemma 5. *The coefficient of x^k in $M_n(x)$, denoted m_k, can be written as*

$$m_k = \sum_{l=0}^{k} \binom{n/2}{l} S(n-l, n-k).$$

Proof. Using the double sum that we have worked out before,

$$M_n(x) = \sum_{i=0}^{n/2} \sum_{j=0}^{n-1-i} \binom{n/2}{i} S(n-i, n-(i+j)) x^{i+j},$$

and by setting $k = i + j$, we get

$$M_n(x) = \sum_{i=0}^{n/2} \sum_{k=i}^{n-1} \binom{n/2}{i} S(n-i, n-k) x^k.$$

We can then change the limits of the inner sum to start at 0 because whenever the second entry of the stirling number is greater than the first, the stirling number will be 0. Changing the limit, we get

$$M_n(x) = \sum_{i=0}^{n/2} \sum_{k=0}^{n-1} \binom{n/2}{i} S(n-i, n-k) x^k.$$

Because the inner sum no longer depends on i, we can change the limits of summation to see

$$M_n(x) = \sum_{k=0}^{n-1} \sum_{i=0}^{n/2} \binom{n/2}{i} S(n-i, n-k) x^k.$$

We are then fixing a specific value of k to find the coefficient of x^k so we can remove the first summation as well as x^k to get

$$m_k = \sum_{i=0}^{n/2} \binom{n/2}{i} S(n-i, n-k).$$

Notice that if $i > k$ then the stirling number in our formula will be 0. Therefore we only need to sum to k. To make this simpler for later use, we can change the sum on i to a sum on l. When we do this we get

$$m_k = \sum_{l=0}^{k} \binom{n/2}{l} S(n-l, n-k).$$

□

This is the last of what we need for the proof of the even case for Theorem 10. We can prove the even case now and then move on to the odd case later. This proof starts by using Theorem 6 and the discussion about separating even $n \times n$ bishop boards. We know that these boards separate into two identical mixed diamond boards. We need to find the rook polynomial for this diamond board and then square it to find the polynomial of the bishop board.

Proof of even case Theorem 10. Let n be even and let B_n be an $n \times n$ board for placing bishops. Also let the coefficient for x^k in $B_n(x)$ be denoted b_k. As we have shown earlier, we can color and split this board into two identical $n \times n-1$ mixed diamond rook sub-boards. When we multiply the resulting rook polynomials together we get the bishop polynomial of B_n. In other words, when n is even,

$$B_n(x) = M_n(x) \cdot M_n(x).$$

In the same fashion as before, we can rearrange the rows and columns of these mixed diamond boards so that they closely resemble a triangle board. We want to look at the rook polynomials for these rearranged boards one coefficient at a time. Consider what we found in Lemma 5 for m_k in $M_n(x)$. We want to find b_m in $B_n(x)$ so we need to multiply the specific coefficients in $M_n(x)$ together so that when we add their corresponding powers of x, we get m. We need $m_s \cdot m_{m-s}$ for all possible values of s. We know that $m_s \cdot m_{m-s}$ is the coefficient of $x^s \cdot x^{m-s} = x^m$ in $B_n(x)$. This leads us to see that

$$b_m = \sum_{s=0}^{m} m_s \cdot m_{m-s}.$$

Now that we have a formula for any coefficient in $B_n(x)$, we can easily sum on the coefficients so that they are multiplied by their corresponding power of x to get $B_n(x)$. This will give us

$$B_n(x) = \sum_{q=0}^{2n-2} \left[\sum_{s=0}^{q} m_s \cdot m_{q-s} \right] x^q.$$

Theorem 7 gives us that the highest coefficient in $B_n(x)$ is $2n-2$ so q only needs to sum to $2n-2$. Expanding both m_s and m_{q-s} using Lemma 5 gives us our final result for even n:

$$B_n(x) = \sum_{q=0}^{2n-2} \sum_{s=0}^{q} \left(\sum_{l=0}^{s} \binom{n/2}{l} S(n-l, n-s) \right) \cdot \left(\sum_{h=0}^{q-s} \binom{n/2}{h} S(n-h, n-(q-s)) \right) x^q$$
$$= \sum_{q=0}^{2n-2} \sum_{s=0}^{q} \left(\sum_{l=0}^{s} \binom{\lceil n/2 \rceil}{l} S(n-l, n-s) \right) \cdot \left(\sum_{h=0}^{q-s} \binom{\lfloor n/2 \rfloor}{h} S(n-h, n-(q-s)) \right) x^q.$$

□

3.2.2 The Polynomial $B_n(x)$ when n is Odd

We will prove the odd case of Theorem 10 using the same argument we used for even n only now we do not have identical sub-boards. Let us first work with the odd diamond boards. Using the same rearrangement technique as seen in Figure 3.10, we can switch the rows and columns again to resemble a triangle board. Recall that the odd diamond board is $n \times n$, so these boards are the same size as the original bishop board. In the same fashion as Lemma 4 we

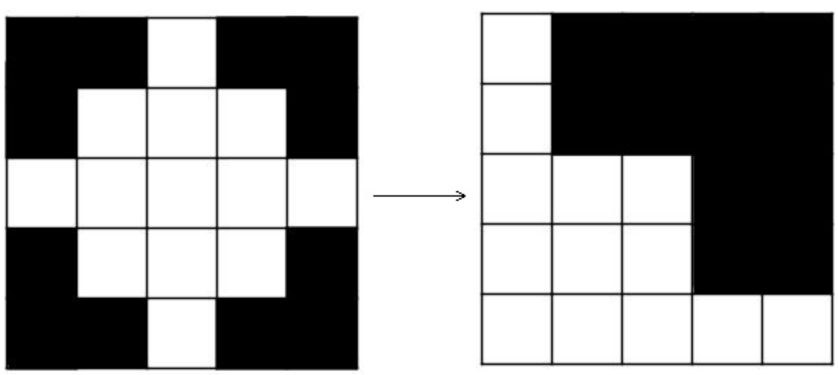

Figure 3.10: Rows and columns of an odd diamond board rearranged.

can find a formula for the rook polynomials for all odd diamond boards. We will denote the rook polynomial for an odd diamond board as $O_n(x)$ and let o_k

be the coefficient of x^k in the rook polynomial for the odd diamond board. We will state the formula for these rook polynomials as a lemma.

Lemma 6. *If n is odd, then*

$$O_n(x) = \sum_{i=0}^{(n+1)/2} \binom{(n+1)/2}{i} T_{n-1-i}(x) \cdot x^i.$$

Proof. This proof will follow the same argument as in Lemma 4 where we are summing over the number of rooks placed on the diagonal. Consider the rook polynomial of the board if we do not use any of the open squares on the diagonal. This is obviously just $T_{n-1}(x)$ as the result of temporarily restricting rooks from the open squares on the diagonal gives us a size $n-1$ triangle board.

Next, we place one rook on the diagonal just as we did before. There are only ever $^{(n+1)}/_2$ open squares on the diagonal so there are $\binom{(n+1)/2}{1}$ ways to choose one of these squares. Notice how in Figure 3.11 when we place a rook on the diagonal and temporarily restrict squares on the diagonal, the open squares form another smaller triangle board again. Because the rook is cancelling a row and column, we can see that the smaller triangle board we get is of size $n-2$. The next term in our rook polynomial must be $\binom{(n+1)/2}{1} \cdot T_{n-2}(x) \cdot x$. Again we are multiplying this polynomial by x because we are placing one rook so we must raise the degree of each term in $T_{n-2}(x)$ by 1.

We continue as we did before, placing more rooks on the diagonal with the remaining open squares "collapsing" into a smaller triangle board. For the last

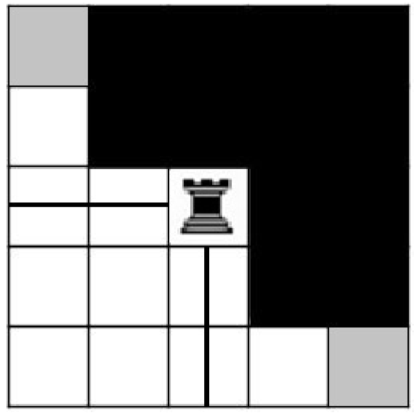

Figure 3.11: Odd diamond board with one rook on the diagonal and other diagonal squares restricted.

term, we look at a rook placement where every square on the diagonal is used. There will be only one way to place these rooks, which is equivalent to saying there are $\binom{(n+1)/2}{(n+1)/2}$ different ways to place them. When we do this, the resulting open squares will collapse again to another smaller triangle board and our final term will be $T_{(n-1-\frac{n+1}{2})}(x) \cdot x^{\frac{n+1}{2}}$. When we combine all these together we get

$$O_n(x) = \binom{(n+1)/2}{0} T_{n-1}(x) + \binom{(n+1)/2}{1} T_{n-2}(x) \cdot x + \cdots + \binom{(n+1)/2}{(n+1)/2} T_{((n-3)/2)}(x) \cdot x^{(n+1)/2}$$

$$= \sum_{i=0}^{(n+1)/2} \binom{(n+1)/2}{i} T_{n-1-i}(x) \cdot x^i.$$

□

We can substitute the formula for triangle boards in terms of stirling numbers using Theorem 9 to find a formula similar to the one we found for the coefficients of mixed diamond rook polynomials. Substituting we get,

$$
\begin{aligned}
O_n(x) &= \sum_{i=0}^{(n+1)/2} \binom{(n+1)/2}{i} T_{n-1-i}(x) \cdot x^i. \\
&= \sum_{i=0}^{(n+1)/2} \binom{(n+1)/2}{i} \left[\sum_{j=0}^{n-1-i} S(n-i, n-i-j) x^j \right] x^i \\
&= \sum_{i=0}^{(n+1)/2} \sum_{j=0}^{n-1-i} \binom{(n+1)/2}{i} S(n-i, n-(i+j)) x^{i+j}.
\end{aligned}
$$

Again, this is a rather difficult sum to work with. We simplify this using our next Lemma.

Lemma 7. *The coefficient of x^k in $O_n(x)$, denoted o_k, can be written as*

$$o_k = \sum_{l=0}^{k} \binom{n+1/2}{l} S(n-l, n-k).$$

Proof. This proof will be identical to that of Lemma 5. In the interest of not being redundant we will not repeat every detail. Notice however that the number of open squares on the diagonal is $^{(n+1)}/_2$. This difference will only change the final formula to give us

$$o_k = \sum_{l=0}^{k} \binom{(n+1)/2}{l} S(n-l, n-k).$$

□

Now we need to work with even diamonds. These are very similar to the odd diamonds with only one small change. Looking at an even diamond board we can recognize one small problem. Start by rearranging the rows and columns in a similar way we did with mixed and odd diamonds. While they resemble triangle boards, they have open squares above the diagonal. Triangle boards do not have any open squares above the diagonal. Also recall that even diamond boards are of the size $n-1 \times n-1$. It turns out there is an easy solution to both of these problems. By adding a row of restricted squares on the top and a column of restricted squares on the right, we now have an $n \times n$ board with no open squares above the diagonal without changing the rook polynomial. An example of this is shown in Figure 3.12. We can now state the

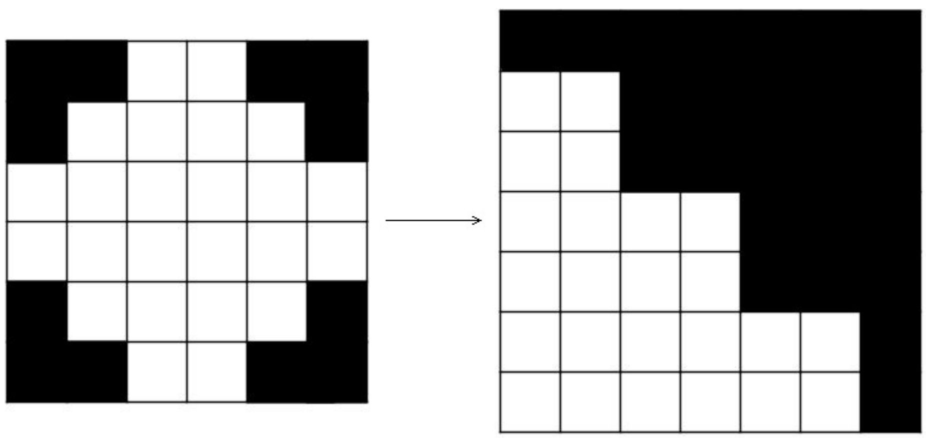

Figure 3.12: Rows and columns of an even diamond rearranged. Extra row and column added to make the board $n \times n$.

formula for the rook polynomial for an even diamond board. We will use $E_n(x)$ to denote the rook polynomial of an even diamond board.

Lemma 8. *If n is odd, then*

$$E_n(x) = \sum_{i=0}^{(n-1)/2} \binom{(n-1)/2}{i} T_{n-1-i}(x) \cdot x^i.$$

Proof. The argument for this proof is exactly the same as for Lemma 4 and Lemma 6 with only one noticeable difference. In the even diamond boards there are only $^{(n-1)}/_2$ open squares on the diagonal. In the interest of not being unneccessarily repetitive, we use the exact same argument with this one difference in mind to conclude that

$$E_n(x) = \sum_{i=0}^{(n-1)/2} \binom{(n-1)/2}{i} T_{n-1-i}(x) \cdot x^i.$$

□

Again we substitute the formula for triangle boards in terms of stirling numbers using Theorem 9 to see that

$$\begin{aligned} E_n(x) &= \sum_{i=0}^{n-1/2} \binom{n-1/2}{i} T_{n-1-i}(x) \cdot x^i. \\ &= \sum_{i=0}^{n-1/2} \binom{n-1/2}{i} \left[\sum_{j=0}^{n-1-i} S(n-i, n-i-j) x^j \right] x^i \\ &= \sum_{i=0}^{n-1/2} \sum_{j=0}^{n-1-i} \binom{n-1/2}{i} S(n-i, n-(i+j)) x^{i+j}. \end{aligned}$$

This leads us to our next lemma. Because the proof for Lemma 9 is identical to that of Lemma 5 and Lemma 7, the proof for Lemma 9 will be omitted.

Lemma 9. *The coefficient of x^k in $E_n(x)$, denoted e_k, can be written as*

$$e_k = \sum_{l=0}^{k} \binom{n-1/2}{l} S(n-l, n-k).$$

This is all we need to prove the odd case of Theorm 10. This proof for the odd case will follow a very similar argument to that of the even case.

Proof of the odd case of Theorem 10. Let n be odd and let B_n be an $n \times n$ board for placing bishops. Also let the coefficient of x^k in $B_n(x)$ be denoted b_k. Because n is odd, when we color and split B_n into sub-boards as we have done before, we end up with an $n \times n$ odd diamond and an $n-1 \times n-1$ even diamond. We can rearrange the rows and columns as we did before to find boards that resemble triangle boards. When rearranging the even diamond, remember that we must add a row and column of restricted squares to make the dimensions $n \times n$. In order to find $B_n(x)$, we need to multiply the rook polynomials of these two together. When n is odd,

$$B_n(x) = O_n(x) \cdot E_n(x).$$

Consider the results of Lemma 7 and Lemma 9 for o_k and e_k. We want to use these results to find b_m. Because of how we defined o_k and e_k, we know that $o_s \cdot e_{m-s}$ is the coefficient of $x^s \cdot x^{m-s} = x^m$ in $B_n(x)$. This leads us to

$$b_m = \sum_{s=0}^{m} o_s \cdot e_{m-s}.$$

Now that we have a formula for any coefficient of $B_n(x)$, we need only sum on the powers of x multiplied by their coefficients. Further recall that from

Theorem 7 that the highest power can only be $2n-2$. Now we have

$$B_n(x) = \sum_{q=0}^{2n-2}\left[\sum_{s=0}^{q} o_s \cdot e_{q-s}\right]x^q.$$

We now can expand o_s and e_{q-s} using Lemma 7 and Lemma 9 to see

$$\begin{aligned} B_n(x) &= \sum_{q=0}^{2n-2}\sum_{s=0}^{q}\left(\sum_{l=0}^{s}\binom{(n+1)/2}{l}S(n-l,n-s)\right)\cdot\left(\sum_{h=0}^{q-s}\binom{(n-1)/2}{h}S(n-h,n-(q-s))\right)x^q \\ &= \sum_{q=0}^{2n-2}\sum_{s=0}^{q}\left(\sum_{l=0}^{s}\binom{\lceil n/2\rceil}{l}S(n-l,n-s)\right)\cdot\left(\sum_{h=0}^{q-s}\binom{\lfloor n/2\rfloor}{h}S(n-h,n-(q-s))\right)x^q. \end{aligned}$$

□

In the even case, the ceiling and floor functions had no effect on the calculation as n was even. When n is odd however, the ceiling and floor functions serve an important purpose as $n/2$ would not give us an integer. Combining this proof of the odd case with the one we found earlier for the even case, we have a full proof for Theorem 10.

		k										
		0	1	2	3	4	5	6	7	8	9	10
n	0	1										
	1	1	1									
	2	1	4	4								
	3	1	9	26	26	8						
	4	1	16	92	232	260	112	16				
	5	1	25	240	1124	2728	3368	1960	440	32		
	6	1	36	520	3896	16428	39680	53744	38368	12944	1600	64

Table 3.2: Number of ways to place k bishops on an $n \times n$ board.

3.3 Maximal Number of Bishops and Bishop Placements on Rectangular Boards

What happens to the number of bishops we can place on a board when it is no longer square? We now have many more boards to consider with a variety of rook sub-boards. Before we begin to discuss what these might look like, we need some new concepts. We start by describing the process of finding the permanent of a matrix.

Definition 11. *The **permanent** of an $n \times n$ matrix A, denoted $per(A)$, can be written*

$$per(A) = \sum_{p \in S_n} a_{1,p(1)} a_{2,p(2)} \cdots a_{n,p(n)},$$

where p is a permutation of the set $\{1, 2, 3, ..., n\}$ [12].

For a 2×2 matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad \text{per}(A) = ad + bc.$$

We build our method of finding the permanent of larger matrices recursively. When we have a larger matrix, we begin by choosing any row. We take the first element in that row, which we will call a, and multiply a by the permanent of the sub-matrix we get when we delete the row and column that a inhabits. Then we do the same thing for our second element and add it to what we found for our first element. Following this process with every element in the row, we will eventually arrive at the value of the permanent. This might sound somewhat familiar to any reader who is has experience with the determinant of a matrix. The process for finding both the permanent and determinant is the same, however the determinant is an alternating sum and the permanent is not. As an example, for a 3×3 matrix we have,

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \Rightarrow \quad \text{per}(B) = a(ei + fh) + b(di + fg) + c(dh + eg).$$

An interesting property of the permanent is seen in the case where we are dealing with a {0,1} square matrix. For our purposes, we define our matrix using permutations with restricted positions. Recall that from our earlier discussion of rook polynomials, a square rook board is a visual representation of a set of permutations with restricted positions. For our square matrix A, we define each entry as

$$a_{i,j} = \begin{cases} 1 & : \text{if } j \text{ is permitted to occupy the } i\text{th place} \\ 0 & : \text{otherwise.} \end{cases}$$

Another way to describe this matrix is in terms of a rook board. If we were to create a matrix from a square rook board in the fashion described above, an open square would correspond to a 1, and a restricted blacked out square would correspond to a 0. For our purposes, the matrix A would be defined in terms of a rook board B by

$$a_{i,j} = \begin{cases} 1 & : \text{if position } (i,j) \text{ in } B \text{ is unrestricted} \\ 0 & : \text{otherwise.} \end{cases}$$

This leads us to a very interesting theorem using the permanent, the proof for which is drawn from Vladimir Baltić [2].

Theorem 11. *The number of permutations with restricted positions is given by the permanent corresponding to a square* $\{0,1\}$ *matrix A.*

Proof. Using Definition 11 for the permanent of a matrix, consider the right hand side. Consider a permutation p that does not fulfill the requirements of our matrix A. We know there is some entry $a_{i,p(i)} = 0$ because of the way we defined each entry. This means that the product $a_{1,p(1)}a_{2,p(2)} \cdots a_{n,p(n)}$ will end up being 0 when we have a permutation that does not fulfill the requirements given by A. This means that the permutations that do not satisfy the requirements given by A will contribute nothing to the sum. Now consider a permutation p that does satisfy the conditions given by A. This means that each entry on the right hand side will be 1. The overall product, $a_{1,p(1)}a_{2,p(2)} \cdots a_{n,p(n)}$, for each permutation that does satisfy the conditions given by A will be 1. This means that when we add all the terms together, we have the number of permutations that satisfy the conditions of our matrix A. □

This is very important to us as this proves that if we were to take any square rook board, and then convert it to a {0,1} matrix using the method we described above, and then find the permanent, we would have the number of ways to place n rooks on the board. When we can place n rooks on the board, this will be our leading coefficient. However, if we have an $n \times n$ board that cannot fit n rooks, like an odd diamond, the permanent will not give us the leading coefficient. Keeping this fact in mind, we move back to rectangular bishop boards. In the investigation of rectangular bishop boards, we notice two meaningful cases. One case is when both m and n are even. This case turns out to be slightly more complicated than our second case of when either m, n, or both m and n are odd.

3.3.1 Rectangular Bishop Boards when m is Odd

We begin by discussing $m \times n$ bishop boards such that m is odd. Without loss of generality, let us start by saying for any board of this type that $m < n$. If $m = 1$, then when we find the rook sub-boards, we have a series of disjoint squares. We can place k bishops on the board in $\binom{n}{k}$ ways. We will now assume that $m > 1$ is odd and look at two sub-cases, the first being when n is even. Looking at an $m \times n$ board where m is odd and n is even, we will follow the same procedure we used when looking at square boards. We color the board and split it into rook sub-boards. When we reflect the colored bishop board over the center row, we have the same board again with opposite coloring. This means that the two colors will always produce the same sub-board. What is interesting about these boards is that they have a slight similarity to the mixed diamonds we saw earlier. However they do not share similar properties.

Definition 12. *For m odd, a **mixed slant board** is a rook board of square dimensions* *that has open squares that follow the pattern*

$$1, 3, 5, ..., m-4, m-2, m, ..., m, m-1, m-3, ..., 6, 4, 2,$$

where the middle rows with m open squares are offset by 1 column from each other forming a slanted shape.

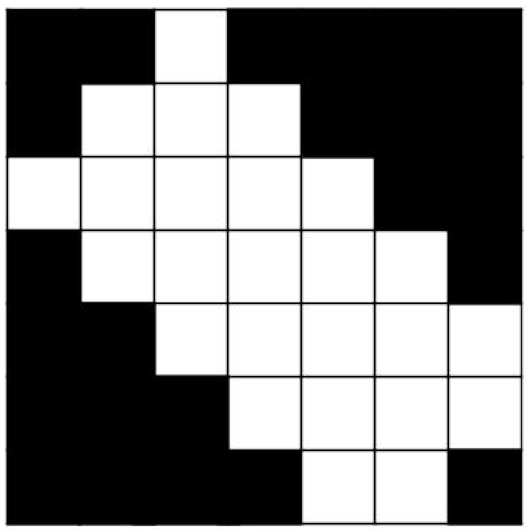

Figure 3.13: A mixed slant sub-board from a 5×10 bishop board.

When we reach the rows with m open squares, they become offset from each other. This is where they differ from our other boards. The number of offset rows of m open squares is dependant on n. Unfortunately, these boards do not have a clean recursive relationship like we had for the triangle boards. Finding a formula for the rook polynomials based on m and n is a much more difficult problem. For now, we will focus only on maximal rook placements. One very important note is that these sub-boards are square. Because these boards are square, we can use Theorem 11 by converting these boards into {0,1} matrices and finding the permanent. As we showed in the proof, as long as we can place a rook in each row and column, the permanent tells us the number of ways to place the maximum number of rooks on the corresponding boards.

Now let us look at the case when both m and n are odd. We no longer have the same kind of symmetry that we had before when n was even. When we color these boards and separate into rook sub-boards, we will have two sub-boards. Our fist sub-board will correspond to the color that we used in the top left of the bishop board.

Definition 13. *For m odd, an **odd slant board** is a board of square dimensions for rook placement that has open squares that follow the pattern*

$$1, 3, 5, ..., m-3, m-2, m, ..., m, m-2, m-3, ..., 5, 3, 1,$$

where the middle rows of m open square are offset 1 column from each other producing a slanted shape.

Much like we had in the mixed slant boards, the middle rows of m open squares are offset from each other. When we choose the other color from our colored bishop board, we get a different kind of rook sub-board.

Definition 14. *For m odd, an **even slant board** is a board of square dimensions for rook placement that has open squares that follow the pattern*

$$2, 4, 6, ..., m-3, m-1, m, ..., m, m-1, m-3, ..., 6, 4, 2,$$

where the middle rows with m open squares are offset 1 column from each other forming a slanted shape.

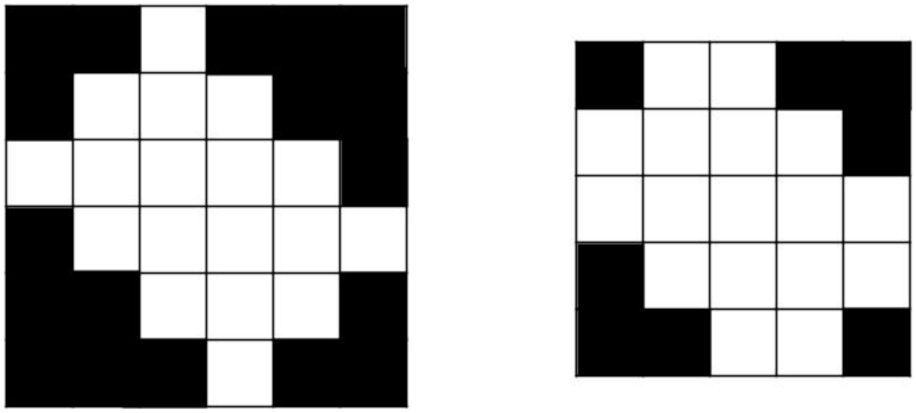

Figure 3.14: The resulting even and odd slant sub-boards from a 5×7 bishop board.

As we can see in Figure 3.14, these sub-boards are not the same. However, they slant in the exact same way as the mixed slant boards. The rows with m open squares are offset in the same way as before. We know that in order to find the number of ways to place the maximum number of rooks on these boards, we need to multiply the permanents of their corresponding matrices together. What interests us now is trying to find the maximum number of rooks. To find this, we need to find the dimensions of the sub-boards of each $m \times n$ bishop board when m is odd.

Lemma 10. *Let B be an $m \times n$ board for bishop placement with m odd, n even and $m < n$. The degree of the polynomial for the resulting mixed slant rook sub-board is*

$$\frac{m-1}{2} + \frac{n}{2}.$$

Proof. We will prove this by induction on n. Consider an $m \times (m+1)$ bishop board. We know that the resulting sub-board will have open squares following the pattern

$$1, 3, 5, ..., m-4, m-2, m, m-1, m-3, ..., 6, 4, 2.$$

Notice that in the first case, there will be no repeated rows of m open squares in the middle. We want to count the number of rows we have in the mixed slant board. We can split these into counting the odd integers up to some odd m which is clearly $\frac{m+1}{2}$. Then we want to count the even integers up to some odd m which is also clearly $\frac{m-1}{2}$. Adding these together we get m rows.

Because we found m rows and we know that mixed slant boards are of square dimensions, we have that the degree of the rook polynomial has m as an upper bound. We need to show that we can place rooks in each row and column on any mixed slant board. Consider the following method for placing rooks on a mixed slant board. Start in the first column and place a rook in the only square there. Moving up and to the right, we place rooks until we reach to the top row. Now moving over to final row, we place a rook in the leftmost open square in the bottom row. Then move up and to the right as we did before, until we have placed rooks all the way up to the rook in the last column on the top open square. What remains after these rook placements is a number of open squares that will contain a section of the diagonal. Place the remaining rooks in this open section along the diagonal. An example of this method is shown in Figure 3.15. Following this method we can see that because we found m rows and columns, we can place m rooks on the board.

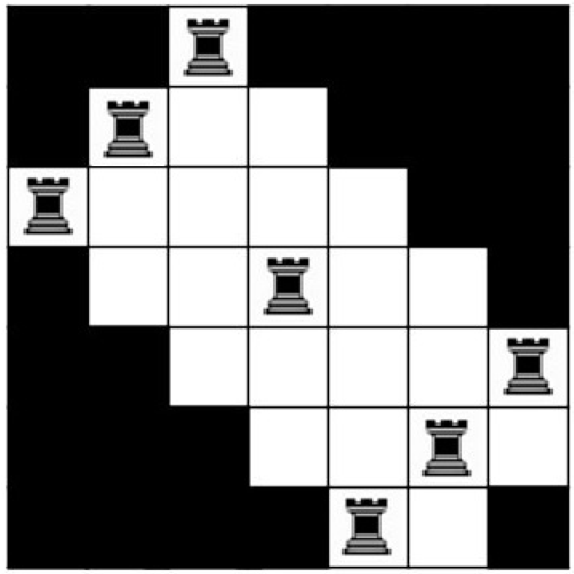

Figure 3.15: Placement of rooks in each row and column of mixed slant board from Figure 3.13.

Plugging into our fomula to show the base case, we get

$$\frac{m-1}{2}+\frac{n}{2}=\frac{m-1}{2}+\frac{(m+1)}{2}=m.$$

We can now assume our lemma is true for an even n and we want to show it is true for $n+2$. Look at an $m \times n+2$ bishop board. More specifically, examine the mixed slant sub-board. Notice that this is exactly the same as a mixed slant sub-board for an $m \times n$ board, but with one extra row of m offset open squares. Because this one extra row is offset, it is adding an extra column as well. When we increase n by 2 then we are adding one row and one column. Therefore we are increasing the degree of the rook polynomial of the smaller board by 1. In order to prove the lemma, we need to show that the degree of the board

$m \times n + 2$ is one more than the degree of the $m \times n$ board. We have that

$$\frac{m-1}{2} + \frac{n+2}{2} = \frac{m-1}{2} + \frac{n}{2} + 1$$

which is exactly the degree of the smaller board plus one. Thus, by induction, we have shown that the degree of the rook polynomial for a mixed slant sub-board of an $m \times n$ bishop board is

$$\frac{m-1}{2} + \frac{n}{2}.$$

□

Lemma 11. *Let B be an $m \times n$ bishop board with $m < n$ and both m, n odd. The degree of the rook polynomial of the resulting odd slant sub-board is*

$$\frac{m-1}{2} + \frac{n+1}{2},$$

and the degree of the polynomial of the resulting even slant sub-board is

$$\frac{m-1}{2} + \frac{n-1}{2}.$$

Proof. This proof is very similar to that of Lemma 10 and will omit some details. For our base case of the odd slant board, we have an $m \times m + 2$ bishop board. This will give us an odd slant board with open squares following the pattern

$$1, 3, 5, ..., m, m, m-2, ..., 5, 3, 1.$$

Counting these rows and columns in a similar fashion as in Lemma 10 we get that there are $m+1$ rows and columns. Consider the following method for rook placement on any odd slant board. Begin as we did for mixed slant boards by placing a rook in the first column and moving up and to the right until we have placed a rook in the top row. Then move to the bottom row and place a rook in the only open square. As before, move up and to the right until a rook is placed in the last column. Again, what remains is a set of open squares that contains the diagonal. Place rooks on every square of the diagonal in this middle open section. An example of this method is shown in Figure 3.16. Now that we can place $m+1$ rooks on the board, we have that

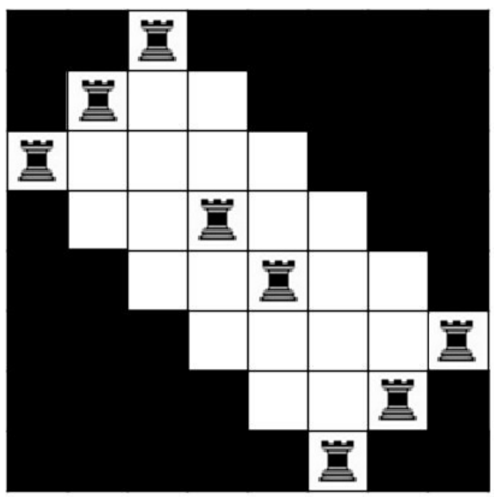

Figure 3.16: A placement of rooks on an odd slant board.

$$\frac{m-1}{2}+\frac{n+1}{2}=\frac{m-1}{2}+\frac{(m+2)+1}{2}=m+1.$$

We can now assume our lemma holds for the odd slant sub-boards of $m \times n$ bishop boards. Again, when we move to the next larger case, we add a row of

m offset squares in the middle, which adds a row and column. This means the degree of the next larger case must just add one to the degree of the smaller case. Our next larger case would be an $m \times n + 2$ board which gives us

$$\frac{m-1}{2} + \frac{(n+2)+1}{2} = \frac{m-1}{2} + \frac{n+3}{2} = \frac{m-1}{2} + \frac{n+1}{2} + 1.$$

We have shown by induction that the degree of the rook polynomial of an odd slant sub-board of an $m \times n$ bishop board is

$$\frac{m-1}{2} + \frac{n+1}{2}.$$

For the even slant sub-board of our $m \times m + 2$ bishop board base case, we know the open squares will follow the pattern

$$2, 4, 6, ..., m-1, m, m-1, ..., 6, 4, 2.$$

As before, we need to count the rows and columns. Using a similar method, we get that there are m rows and columns in the even slant board. We use a similar method of placing rooks as before to show that we can always fit rooks in each row and column. Start in the lowest open square in the first column and move up and to the right until a rook is placed in the top row. Then, we go to the left square in the bottom row, and move up and to the right until we reach the last column. However, if there is a column of all open squares, we must start in the right square of the bottom row. This case does not occur often, but should be mentioned. What remains is again the section containing the diagonal so we place rooks on the diagonal as before to cover the remaining rows and columns. An example is shown in Figure 3.17. Now that

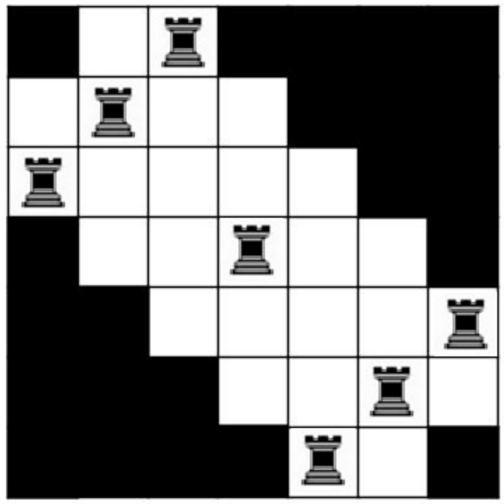

Figure 3.17: A placement of rooks on an even slant board.

we know we can place m rooks on the board, we have that

$$\frac{m-1}{2} + \frac{n-1}{2} = \frac{m-1}{2} + \frac{(m+2)-1}{2} = m.$$

We can assume this is true for the even slant sub-boards of $m \times n$ bishop boards. Again, when we move to the next larger case, we add a row of m offset squares in the middle, which adds one row and one column. Our next larger case gives us an $m \times n + 2$ bishop board. We want to show that this is one more than the degree of the even slant sub-board of an $m \times n$ bishop board. We see that

$$\frac{m-1}{2} + \frac{(n+2)-1}{2} = \frac{m-1}{2} + \frac{n+1}{2} = \frac{m-1}{2} + \frac{n-1}{2} + 1.$$

We have shown by induction that the degree of the rook polynomial of an even slant sub-board of an $m \times n$ bishop board is

$$\frac{m-1}{2} + \frac{n-1}{2}.$$

□

Theorem 12. *Let B be an $m \times n$ bishop board where m is odd and $m < n$. The maximum number of rooks that can be placed on B is equal to $m + n - 1$.*

Proof. Let us begin with n even. We know that this type of bishop board will give us two identical mixed slant rook sub-boards. As we found in Lemma 10, the degree of the rook polynomials for these sub-boards is

$$\frac{m-1}{2} + \frac{n}{2}.$$

When we multiply these rook polynomials together, we add the degrees to get

$$\frac{m-1}{2} + \frac{n}{2} + \frac{m-1}{2} + \frac{n}{2} = m + n - 1.$$

When n is odd, we use Lemma 11 for the degrees of the resulting even and odd slant rook sub-boards. When we multiply the polynomials together to get the bishop polynomial, we get that

$$\frac{m+1}{2} + \frac{n-1}{2} + \frac{m-1}{2} + \frac{n-1}{2} = m + n - 1.$$

This means that for any $m \times n$ rook board when m is odd and $m < n$, we will always be able to place $m + n - 1$ rooks on the board.

□

The number of ways to place these bishops do not follow any clean pattern. The simplest way to find the number of bishops is to split the board into its corresponding rook sub-boards, convert them to {0,1} matrices, find the permanents, and multiply them together.

3.3.2 Rectangular Bishop Boards when m is Even

We will now look at bishop boards that are $m \times n$ where m is even. Again without loss of generality we will assume that $m < n$. Because m is even, we will always have the property that both rook sub-boards will be identical no matter the parity of n. We can reflect across the middle row and have the same but opposite coloring just as we did before when we were dealing with m odd and n even. Let us start by looking at the case when n is odd.

Definition 15. *For m even, a* ***mixed oblique board*** *is a rook board of square dimension that has open squares that follow the pattern*

$$1, 3, 5, ..., m-1, m, ..., m, m-2, ..., 6, 4, 2,$$

where the middle rows of m squares are each offset by 1 column and create a slanted shape.

A mixed oblique board is similar to the mixed slant board, but only occurs when m is even. The difference is slight but important. When we split an $m \times n$ bishop board where m is even and n is odd, we get two identical mixed oblique sub-boards. As before when dealing with odd m, we do not have a nice recursive relationship to break down these boards. Therefore we will

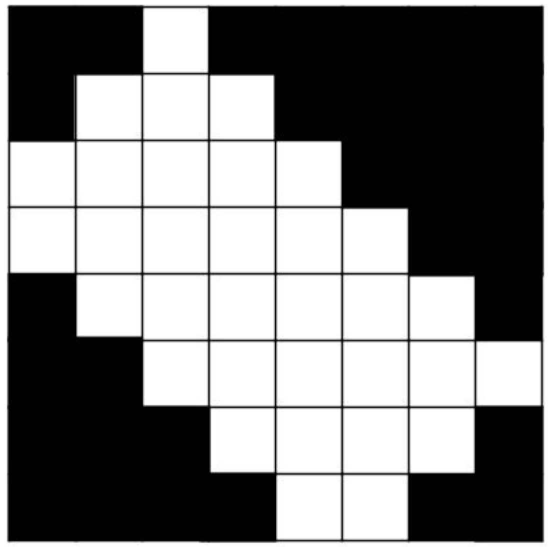

Figure 3.18: The mixed oblique board resulting from a 6×11 bishop board.

focus on leading terms of the various rook and bishop polynomials. In order to find the leading coefficient of each bishop polyomial, we must use our permanent of a matrix method using Theorem 11 for the rook sub-boards and multiply the results.

Lemma 12. *Let B be an $m \times n$ bishop board with m even, n odd, and $m < n$. The degree of the rook polynomial for the resulting mixed oblique sub-board of B is*

$$\frac{n+1}{2} + \frac{m-2}{2}.$$

Proof. We will prove this in a similar fashion to Lemmas 10 and 11. Our base case will be an $m \times m + 1$ bishop board. We can count the rows and columns of the resulting mixed oblique sub-board in a similar way that we counted the mixed, even, and odd slant boards. We find that there are m total rows and columns. We need to show that we can place rooks in every row and column of any mixed oblique board. Consider this method of placing rooks. Start by placing a rook in the top square of the first column. Then move up and to the right until you place a rook in the top row. Then go to the bottom row and place a rook in the right square. Move up and to the right again until you place a rook in the last column. Then place rooks along the remaining diagonal section just as before. An example of this placement is shown in Figure 3.19. Now that we know we can place m rooks on this board, we know

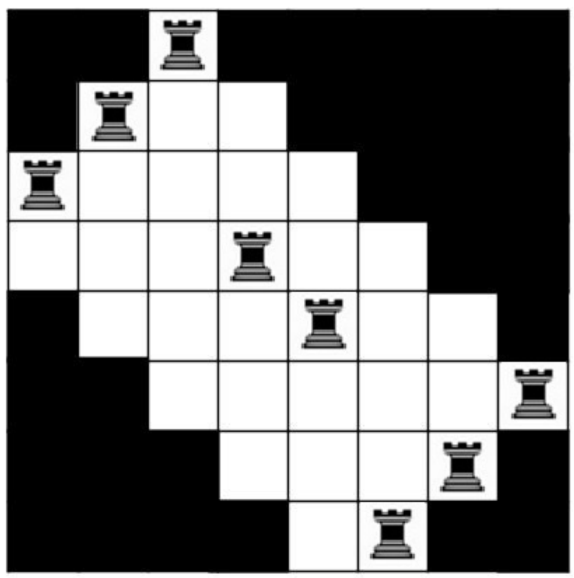

Figure 3.19: A placement of rooks on a mixed oblique board.

the degree of the rook polynomial for our base case mixed oblique sub-board must be m. Using our formula we see that

$$\frac{n+1}{2} + \frac{m-2}{2} = \frac{(m+1)+1}{2} + \frac{m-2}{2} = m.$$

We can now assume this lemma is true for an $m \times n$ board. When we move to the next larger case, we get an $m \times n + 2$ board. We again are adding a row and column due to the extra row of offset m squares. When we move to the next larger case we need to increment the degree we assumed to be true from our formula by one. Using our formula we see

$$\frac{(n+2)+1}{2} + \frac{m-2}{2} = \frac{n+3}{2} + \frac{m-2}{2} = \frac{n+1}{2} + \frac{m-2}{2} + 1.$$

Because this is the formula for an $m \times n$ board plus 1, we have shown through induction that the degree of the rook polynomial for an mixed oblique sub-board of an $m \times n$ bishop board is

$$\frac{n+1}{2} + \frac{m-2}{2}.$$

□

Theorem 13. *Let B be an $m \times n$ bishop board with m even, n odd, and $m < n$. We can place at most $m + n - 1$ non-attacking bishops on B.*

Proof. Split B into its corresponding rook sub-boards, to produce two identical mixed oblique rook boards. In order to find the bishop polynomial of B we multiply the rook polynomials of the sub-boards together. In order to find the degree of the bishop polynomial we add the degrees of the rook polynomials from the sub-boards. We found the degrees for the mixed oblique sub-boards in Lemma 12. Adding them together we get

$$\frac{n+1}{2} + \frac{m-2}{2} + \frac{n+1}{2} + \frac{m-2}{2} = (n+1) + (m-2) = m + n - 1.$$

□

This result turns out to be the same as when m was odd. This is not necessarily surprising. When we have either m, n, or m and n odd, our bishop polynomial will be of degree $m + n - 1$.

Now let us look at what happens when both m and n are even. We get an interesting rook sub-board.

Definition 16. *For m even: An **Odd Oblique Board** is a rook board that has open squares that follow the pattern*

$$1, 3, 5, ..., m - 1, m, ..., m, m - 1, ..., 5, 3, 1,$$

where the middle rows of m squares are offset 1 column and create a slant shape.

It is important to note that an odd oblique board does not have square dimensions.

Unfortunately, because the odd oblique rook sub-boards are not square boards, we need to modify our definition of the permanent to count the number of ways to place the maximum amount of rooks because we only proved the theorem for square matrices. There is an extension of the permanent that will work for non-square matrices that will count the number of ways to place the maximum amount of rooks. Say our odd oblique board has dimensions $k \times k - 1$. This means the upper bound on the number of rooks we can place is $k - 1$. Before when we defined the permanent we used a permutation p. We can just redefine the permanent of any $m \times n$ matrix where $n > m$ with $p \in P(m, n)$ where p is an m-permutation of elements from the set

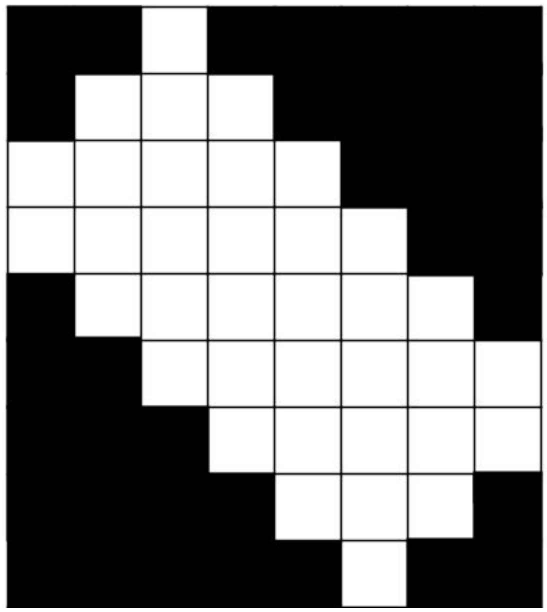

Figure 3.20: The odd oblique sub-board from a 6×12 bishop board.

$\{1, 2, 3, ..., n\}$ [9]. As an example, a 4-permutation of the set $\{1, 2, 3, 4, 5\}$ is $\{3, 4, 5, 1\}$. We simply choose m elements from the set of n elements and permute them. If we can place m rooks on the corresponding board, then this definition of the permanent will tell us how to place those m rooks.

We can look at the degree of the rook polynomial for odd oblique boards and find the degree of the bishop polynomial for m and n both even.

Lemma 13. *Let B be an $m \times n$ bishop board with m, n even, and $m < n$. The degree of the rook polynomial for the resulting odd oblique sub-board of B is*

$$\frac{n}{2} + \frac{m-2}{2}.$$

Proof. This proof will work in a similar way as our other similar lemmas, but we need to show that we can place rooks only in each column and not each row and column as the odd oblique boards do not have square dimensions. Our base case is an $m \times m + 2$ bishop board. We split this board into the two identical odd oblique sub-boards. We need to count the number of columns we get. If we count the rows as we did in the proof of Lemma 10, we get $m + 1$ rows. However, we remember that all odd oblique boards have one less column than the number of rows. We then have m columns and the upper bound on the number of rooks we can place is m. We now need to show a method that will always guarantee a rook in each column for every odd oblique board. Start by placing a rook in the top square of the first column. Then, as we did before, move up and to the right until a rook is placed in the top row. Next, place a rook in the bottom row and move up and to the right again until a rook is placed in the last column. The squares that remain will have a diagonal that we will place our remaining rooks. An example is shown in Figure 3.21. Now that we know we can place a rook in each of the m

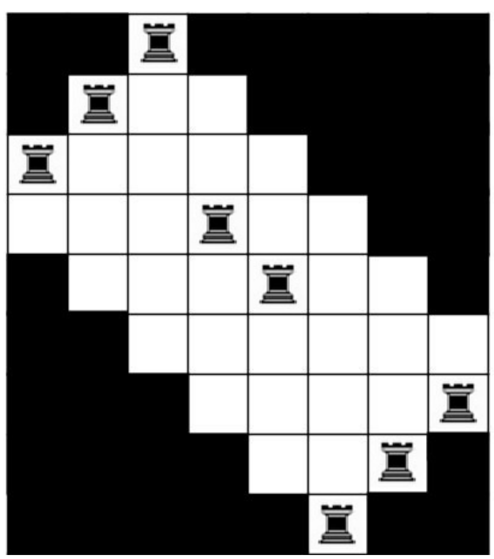

Figure 3.21: A placement of rooks in each column for an odd oblique board.

columns, we know the degree of the rook polynomial must be m. Checking with our lemma we see that

$$\frac{n}{2} + \frac{m-2}{2} = \frac{m+2}{2} + \frac{m-2}{2} = m.$$

Now we can assume our lemma holds for $m \times n$ bishop boards. We want to show that for $m \times n+2$ bishop boards, we have one more than the degree of an $m \times n$ bishop board. Using an $m \times n+2$ board gives us

$$\frac{n+2}{2} + \frac{m-2}{2} = \frac{n}{2} + \frac{m-2}{2} + 1.$$

By induction, we have shown that the degree of an odd oblique sub-board of an $m \times n$ bishop board is

$$\frac{n}{2} + \frac{m-2}{2}.$$

□

Theorem 14. *Let B be an $m \times n$ bishop board where both m, n are even and $m < n$. We can place at most $m + n - 2$ bishops on B.*

Proof. We know that when we split B into rook sub-boards, we get two identical odd oblique boards. When we find the bishop polynomial of B, we multiply the rook polynomials of the sub-boards. To find the degree of the bishop polynomial we then need to add together the degrees of the rook polynomials of the sub-boards. When we add these together, we get

$$\frac{n}{2} + \frac{m-2}{2} + \frac{n}{2} + \frac{m-2}{2} = m + n - 2.$$

□

Now that we have shown the maximum number of bishops we can place on any $m \times n$ board, let us look at some results. Table 3.3 shows the number of ways to place the maximum number of bishops on some $m \times n$ bishop boards. These results were compiled using an online rook polynomial calculator [6].

		n										
		1	2	3	4	5	6	7	8	9	10	11
m	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	4	1	9	1	16	1	25	1	36	1
	3	1	1	8	1	3	4	5	9	16	25	39
	4	1	9	1	16	1	64	9	400	25	1521	81
	5	1	1	3	1	32	1	9	25	75	144	285
	6	1	16	4	64	1	64	1	729	64	10186	625

Table 3.3: Number of ways to place the maximum number of bishops on an $m \times n$ board.

Chapter 4

Rook Boards of Three Dimensions

Now that we have examined rooks and bishops on two-dimensional boards, we will now spend some time looking into extending these theorems into three dimensions. Some research has already been conducted in papers written by Benjamin Zindle [13] and Feryal Alayont [1]. They agree that there are two interpretations to approaching rooks on three-dimensional boards. Before discussing these two interpretations, let us first discuss what a board in three dimensions might look like. An $m \times n \times p$ rook board is essentially p copies of a two dimensional $m \times n$ rook board that are stacked on top of each other. While our interpretation of how rooks can attack will change, our definition of a rook polynomial will not change. We will refer to positions for rook placement as *cubes*.

Definition 17. *Any and all open cubes in a three-dimensonal rook board that have the same first coordinate belong to the same **slab**, open cubes that share the same second coordinate belong to the same **wall**, and open cubes that share the same third coordinate belong to the same **layer**.*

Definition 18. *Any and all open cubes in a three-dimensional rook board that have the same second and third coordinate belong to the same* ***row****, open cubes that share the same first and third coordinate belong to the same* ***column****, and open cubes that share the same first and third coordinate belong to the same* ***tower****.*

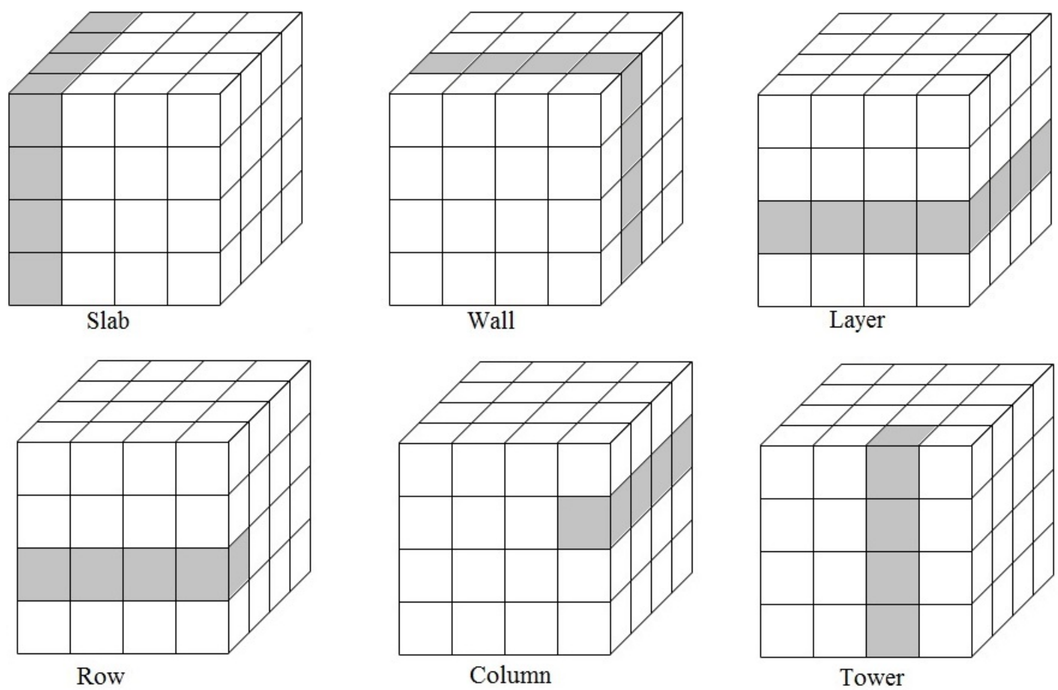

Figure 4.1: A visual example of Definitions 17 and 18.

The differences in these two interpretations stem from the way we allow rooks to attack. The first interpretation allows rooks to attack along planes. Instead of a rook being able to attack along rows and columns, a rook can attack along slabs, walls, and layers. This first interpretation, while mathematically easier to work with, does not fit from our chess move illustration. It extends even easier to boards of many dimensions, as rooks need not attack along lines, but hyperplanes. The second interpretation follows more from our understanding of chess. We still allow rooks to attack along rows and columns just as they would before, but now they can attack up and down in their tower. Zindle and Alayont focused their efforts on the first interpretation. We will look into the second interpretation because in three dimesnions it seems to follow more from our understanding of chess and rook movement.

We will begin by looking at some properties of these new boards that we hope will extend from two dimensions. When dealing with two-dimensional rook boards, one of the most fundamental properties was the ability to exchange rows and columns without altering the rook polynomial.

Theorem 15. *Let B be an $m \times n \times p$ rook board. Rearranging the walls, slabs, or layers will not alter the rook polynomial of B.*

Proof. We want to show that if we rearrange the walls, slabs, and layers, rooks can still only attack the same cubes. Consider a rook placed on an $m \times n \times p$ rook board, B. Because of the way we are allowing rooks to attack, we know that a rook can attack any cube that shares at least 2 of the same coordinates of the position of the rook. Lets say we placed this rook in the location (s, w, l). We want to show that the rook can still attack the cubes in its row, column, and tower. We will start by looking at the row, which means every cube with second and third coordinate w and l respectively. Because rearranging the slabs did not affect the second or third coordinates of any cube, our rook can still attack the same cubes in its original row. If we examine our definition of column and tower, we can see that our rook's column and tower are both contained in the rook's slab, so it will still be able to attack all the same cubes as they moved when we rearranged the slabs. In a similar way, we can use the same argument for exchanging walls and layers. Because changing walls, slabs, and layers does not change which cubes a rook can attack, then it will not affect the rook polynomial. □

Now that we know we can exchange slabs, walls, and layers without changing the rook polynomial, we can start to look at extending Theorem 2 which deals with finding the rook polynomial or separable rook boards.

Theorem 16. *Let B be a three dimensional rook board which can be partitioned into two parts that either share no slabs and walls, walls and layers, or slabs and layers. We will call these partitions C and D. Then*

$$R_B(x) = R_C(x) \cdot R_D(x).$$

Proof. Consider an open cube in C, and without loss of generality, we will assume that open cubes in C share no slabs or walls with D. By the way we allow rooks to attack in our three-dimensional extension, rooks must share at least two coordinates with all cubes they can attack. Because any two cubes with one in C and the other in D do not share any slabs or walls, then they will never have the same first or second coordinate. This means they may only have at most the third coordinate in common. No rook placed in C will be able to attack any cube in D. Using the same argument as we did in Theorem 2, we see that our theorem is true. Our argument would be the same if we have chosen C and D to share no walls and layers, or slabs and layers. □

We have one more theorem from our investigation into two-dimensional rook polynomials that extends cleanly into three dimensions. Theorem 3 has an extension into three dimensions using our interpretation of attacking rooks, however it is much less useful in three dimensions.

Theorem 17. *Let B be a three-dimensional rook board, and let s be one particular open cube of that board. Then let B_1 be the board obtained from B by blacking out the cube s and let B_2 be the board obtained from B by restricting the entire row, column, and tower containing s. Then*

$$R_B(x) = R_{B_1}(x) + x \cdot R_{B_2}(x).$$

Proof. We want to show that every coefficient of x^k is the same on both sides of the equation. The coefficient of x^k in $R_B(x)$ = (Number of ways to place k rooks on B when a rook is not placed in s) + (Number of ways to place k rooks on B when a rook is placed in s) = (Number of ways to place k rooks on B_1) + (Number of ways to place $k-1$ rooks on B_2) = (coefficient of x^k in $R_{B_1}(x)$) + (coefficient of x^{k-1} in $R_{B_2}(x)$) = (coefficient of x^k in $R_{B_1}(x) + x \cdot R_{B_2}(x)$). We can see that the coefficients are the same for any arbitrary k. □

The proof for Theorem 17 is identical to its corresponding two-dimensional proof for Theorem 3. These theorems extend from two-dimensional rook boards to three-dimensional rook boards very clearly. However, not all theorems are quite as obvious. For example, the proofs of Theorems 4 and 5 do not extend cleanly however, the theorems themselves might. Their proofs require a knowlege that when we place a rook, no other rook can be placed in any cube with at least one coordinate in common. These theorems will extend rather cleanly when dealing with Zindle and Alayont's interpretation, but because of how we have interpreted attacking rooks, they will not extend well to our interpretation of attacking rooks.

The main motivation for our investigation into rook polynomials on boards of three dimensions is to find a formula for the number of ways to place the maximum number of rooks on an $n \times n \times n$ board. Before we discuss why this is more difficult than it seems, let us look at some smaller cases. Instead of an $n \times n \times n$ board to start, let us look at an $n \times n \times 2$ board. This is basically just 2 $n \times n$ boards stacked on top of each other.

Lemma 14. *The leading term of the rook polynomial of any unrestricted $n \times n \times 2$ rook board is equal to*

$$n! \cdot D_n \cdot x^{2n},$$

where D_n is the number of derangements of a set of size n.

Proof. Consider placing rooks on the two layers separately starting in the bottom layer. We know that we can place at most n rooks on this board in $n!$ different ways. As we have shown in Theorem 15, we can rearrange the slabs and walls. Let us rearrange them so that all the rooks are on the diagonal of the bottom layer. When we begin to consider placing rooks on the top layer, we cannot place any along the diagonal, as all our rooks on the bottom layer would be able to attack them. As we saw in Example 4, the leading coefficient in the rook polynomial for a board with its diagonal restricted is D_n. We can clearly place n rooks on the upper layer as they are derangements of length n.

Multiplying we see

$$(n! \cdot x^n) \cdot (D_n \cdot x^n) = n! \cdot D_n \cdot x^{2n}.$$

□

n	0	1	2	3	4	5	6	7	8	9
$n! \cdot D_n$	1	0	2	12	216	5280	190800	9344160	598066560	133496

Table 4.1: Maximal Rook Placement on Two Layer Boards.

Having just two layers is pretty simple, however, once we add another layer, it gets much more complicated. We can no longer just rearrange the slabs and walls to get something easy to work with as our third layer depends on the specific choice of our first and second layers. It stands to reason that it gets even more complicated as we get to n layers. It turns out, the leading coefficient of the rook polynomial of an $n \times n \times n$ board is equal to the number of ways to make a size n latin square.

Definition 19. *A **Latin Square** of size n is a grid with n rows and n columns. Each row and column contains the same set of n elements with no repetition of any element in any row or column.*

Look at each individual element in the latin square shown in Figure 4.2. Every element by itself gives a non-attacking rook placement because no element is repeated in any row or column. We know this will always be the case for every element in the latin square. If we were to stack these rook placements given by our latin square on top of each other, where each rook placement given by element k is in layer k, we will have a non-attacking placement of rooks in an $n \times n \times n$ rook board. Clearly, we can only fit at most

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Figure 4.2: One example of a 5×5 latin square.

n rooks on each of the n layers, so our maximal number of rooks we can place on an $n \times n \times n$ board is n^2. Finding the number of ways to create size n latin squares is equivalent to finding the number of ways to place n^2 rooks on an n cube board. Once the connection is made to latin squares, our problem of finding the number of ways to place n^2 rooks on the size $n \times n \times n$ board becomes much more difficult than it seems. There are some formulas to find the size of Latin Squares, however they are not easy to use. There are also upper and lower bounds, but they are not tight, growing apart as n becomes large [12]. The following table shows the number of ways to make latin squares based on n. As of now, only the first 11 values of n are known. These numbers were drawn from sequence A002860 in the OEIS database [10]. The theorems that extend from two dimensions prove to be very inefficient for trying to find a formula for rook polynomials of cube-shaped boards. We started by trying to use Theorem 17 over and over to compute the rook polynomial by breaking down the cube into smaller shapes. However, even if $n = 4$, this proves to be a very long and involved computation. The problem with this theorem is that when we delete the row, column, and tower of any

n	Latin Squares size n
1	1
2	2
3	12
4	576
5	161280
6	812851200
7	61479419904000
8	108776032459082956800
9	5524751496156892842531225600
10	9982437658213039871725064756920320000
11	776966836171770144107444346734230682311065600000

Table 4.2: Number of ways to make a size $n \times n$ Latin Square.

initial cube s, we are left with a very strange shape. It would be much easier had this theorem been able to delete rows, columns, and towers so that we get a smaller more recognizeable shape. If this were the case, we could do something similar to what we did for bishops and prove the formula recursively. If we consider deleting any cube s from a size n cube board, then we can rearrange the walls, layers, and slabs to make s a corner cube. When we delete its row, column, and tower, we end up with a cube-like shape with a tripod-like shape removed. Now we have two cube like shapes that do not have obvious rook polynomials. Then we would have to use Theorem 17 again on both shapes. We would have to do this repeatedly until we reached some shape for which we already knew the rook polynomial, or that we could separate and use Theorem 16. As n gets large, this becomes incredibly time consuming and impractical. There is no obvious way to use these theorems to break down a cube size n into a cube size $n - 1$. If we could, we would have a recursive relationship. Without finding a clean way to reach a recursive relationship, this approach was abandoned. Figure 4.3 shows what will happen to our board when we use Theorem 17.

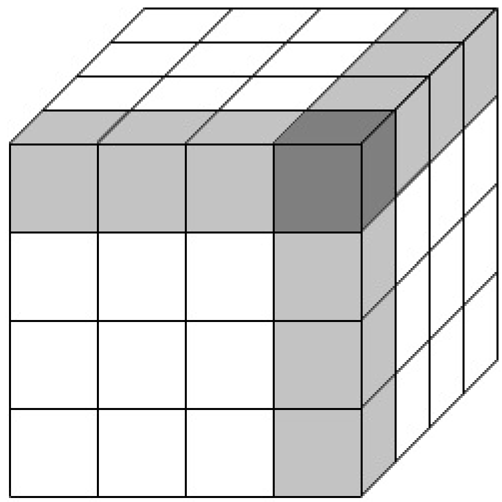

Figure 4.3: When we select the dark gray cube for s, we break into two sub-boards. One with just the dark cube removed, and the other with all the gray cubes removed. Neither resulting sub-board is easy to work with.

Trying to find a recursive relationship in this way makes it very clear why Zindle and Alayont decided not to use our interpretation for attacking rooks. In their interpretation, Theorem 17 would look slightly different. Instead of merely deleting the row, column, and tower, we would be deleting the entire slab, wall, and layer. Once one of each of these is deleted, then we have a smaller cube. However, this would not be helpful in trying to find the number of latin squares. Alayont's version of Theorem 17 with his own interpretation of attacking rooks is given without a proof in Theorem 18 [1].

Theorem 18. *Let B be a board and s be a cube in B, B' be the board obtained by removing the slab, wall, and layer that correspond to s from B, and B'' be the board* *obtained by removing s from B. Then*

$$R_B(x) = x \cdot R_{B'}(x) + R_{B''}(x).$$

In his theorem, we could break down a large cube into a smaller one every time we remove a wall, slab, and layer. This would make the recursive relationship much easier to reach.

Chapter 5

Conclusion

Rook polynomials are not just interesting for their own sake. They have a variety of applications because they directly relate to permutations with restricted positions. This means that rook polynomials can be used in everything from cryptography to combinatorial design theory. Sudoku puzzles are one example of a famous object that can be related to non-attacking rook placements. Keep in mind that sudoku puzzles are a special set of latin squares, a structure to which we have already drawn a connection.

While researching rook polynomials, we can point to many directions for further study. One idea is to investigate boards of higher dimensions. However, as seen in the brief discussion on boards of three dimensions, this becomes difficult rather quickly. It would be interesting to try and find a recursive relationship between cube boards in order to find a general formula for cube rook boards. However this could be difficult to accomplish.

One direction to go would be developing a general formula for the bishop polynomial of an $m \times n$ bishop board using the modified definition of a matrix permanent to find the number of ways to place k bishops on the board. This is a very natural extension from our chapter on bishops. The main difficulty here is that this would not shorten the computation time for the bishop polynomial by a significant amount.

Another interesting avenue for further study would be to put bishops on three-dimensional boards. This leads to many different interpretations as we would need to carefully define how bishops can attack on these boards. There are a few options that would follow from our two-dimensional investigation. We could allow them to attack only on a strict corner diagonal, or attack along any diagonal in its wall, slab, or layer. There is potential for both of these interpretations and it is unclear which would be easier to work with.

We could also examine what would happen to rook and bishop polynomials by changing the shape of the boards. It would be an interesting topic to change the board from a rectangle into a torus. We could create some style of spherical chess board and try and find generating functions for rooks and bishops. We would analyze how the topology of the board would affect the rook polynomial. This would be a very interesting question to persue in further research.

Finally, a very natural extension from the work in this independent study would be examining the remaining chess pieces. Analyzing queens would be the most natural choice for further investigation. Queens can attack both along diagonals and rows and columns. In effect it would be the same as placing a rook and bishop in the same square. It makes sense then that if a non-attacking rook placement is identical to a non-attacking bishop placement on the same board, it is also a non-attacking queen placement. Based on the work we have done using rooks and bishops, it would not be too difficult to gain traction on queen counting problems.

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