2013

The Truth About Lie Symmetries: Solving Differential Equations With Symmetry Methods

Ruth A. Steinhour
The College of Wooster, rasteinhour@gmail.com

The Truth About Lie Symmetries

Solving Differential Equations With Symmetry Methods

Independent Study Thesis

Presented in Partial Fulfillment of the Requirements for
the Degree Bachelor of Arts in the
Department of Mathematics and Computer Science
at The College of Wooster

by
Ruth Steinhour
The College of Wooster
2013

Advised by:

Dr. Jennifer Bowen and Dr. Robert Wooster

THE COLLEGE OF

WOOSTER

Abstract

Differential equations are vitally important in numerous scientific fields. Oftentimes, they are quite challenging to solve. This Independent Study examines one method for solving differential equations. Norwegian mathematician Sophus Lie developed this method, which uses groups of symmetries, called Lie groups. These symmetries map one solution curve to another. They can be used to determine a canonical coordinate system for a given differential equation. Writing the differential equation in terms of a different coordinate system can make the equation simpler to solve. This I.S. explores techniques for finding a canonical coordinate system and using it to solve a given differential equation. Several examples are presented.

Acknowledgments

First and foremost I would like to thank Professor Jennifer Bowen and Professor Robert Wooster for their advice and support. I would also like to thank my parents, Sarah and Wayne Steinhour, my sister, Elizabeth Steinhour, and my friends, both on campus and off, for their continued encouragement.

Contents

CHAPTER 1

Introduction and Symmetries

1.1 Introduction

Differential equations are used to model numerous phenomenon in our world, from the spread of infectious diseases to the behavior of tidal waves. Naturally, the study of differential equations plays a vital role in the physical sciences. These equations are often non-linear and solving them requires unique and creative methods.

This Independent Study will explore the use of Lie symmetry groups for solving differential equations. Sophus Lie (pronounced Lee), a Norwegian mathematician born in 1842, developed these techniques [3]. He was the first to discover the relationship between group theory and traditional methods for finding solution curves [3]. His ground-breaking discovery involved using groups of point transformations to solve differential equations [3].

This IS is the culmination of a two-semester exploration of Lie symmetry groups and their role in solving differential equations. Chapters 1, 2, and 3 explore first order differential equations. In Chapter 1, we define some basic terminology and familiarize the reader with a few examples. Chapter 2 includes a variety of examples that illustrate how Lie Symmetries are utilized to solve both linear and non-linear first order differential equations. The third chapter will define the infinitesimal generator and discuss its role. The information presented in Chapter 3 can be extended for working with higher order differential equations. Chapter 4 discusses higher order differential equations and uses Lie Symmetry groups in a preliminary example.

1.2 Symmetries

The information in this chapter is adapted from Hydon's *Symmetry Methods for Differential Equations: A Beginner's Guide* [2] and Starrett's "Solving Differential Equations by Symmetry Groups" [5]. Material presented includes an introduction to symmetry and Lie groups, along with preliminary examples. These examples come from both sources.

The properties of symmetries provide a unique tool for solving differential equations. A *symmetry* is a rigid mapping from an object to itself or another object. It must preserve the structural properties of the original item. These mappings include rotations, translations, and reflections. One simple example is S_3, the group of symmetries of a triangle. A *group* is a set that is closed under a binary operation. Groups are associative under the operation, contain an identity element, and inverses. The group S_3 is a group of permutations that is closed under composition. Figure 1.1 demonstrates one of these rotations. The permutation is denoted (123). This means that 1 is mapped to 2, 2 is mapped to 3 and 3 is mapped to 1. Notice that the mapping preserves the structure of the triangle.

This demonstrates a *discrete symmetry*. The group S_3 has 6 elements - 6 possible symmetries of an equilateral triangle. This IS deals with *Lie symmetries*, which are

Figure 1.1: Rotating symmetry of a triangle

not discrete.

Symmetries preserve structure. In other words, they must satisfy the *symmetry condition*. In the previous example, satisfying the symmetry condition means that the mapping is a bijection from an equilateral triangle to itself. Applied to the solution curves of a differential equation, satisfying the symmetry condition requires that a symmetry maps one solution curve to another. The resulting solution curve must also satisfy the original differential equation. The symmetry condition is crucial to solving differential equations with symmetry methods. We will discuss this in greater detail in Section 1.4.

As a simple example of symmetry in differential equations, consider the following ordinary differential equation (ODE),

$$\frac{dy}{dx} = 0. \tag{1.1}$$

As one would expect the solutions to this equation are horizontal lines, seen in Figure 1.2.

Lie symmetries are *point transformations* [6] that map a point on a solution curve in

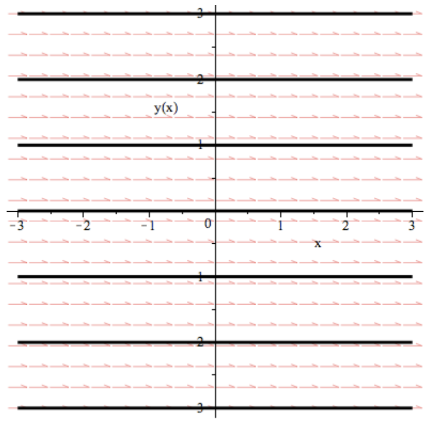

Figure 1.2: Solutions to Equation (1.1)

$\mathbb{R}^2$ to another point on a solution curve in $\mathbb{R}^2$. For a parameter $\lambda \in \mathbb{R}$, a Lie symmetry looks like:

$$P_\lambda : (x, y) \mapsto (\hat{x}, \hat{y}).$$

The identity, I in a Lie symmetry group maps a point to itself.

For example, one symmetry of Equation (1.1) is

$$(\hat{x}, \hat{y}) = (x, y + \lambda). \tag{1.2}$$

This symmetry will map a point (x, y) on one solution curve to a point $(\hat{x}, \hat{y})$ on another. Some symmetries are trivial. A *trivial symmetry* maps every solution curve to itself. For instance, a trivial symmetry of Equation (1.1) is:

$$(\hat{x}, \hat{y}) = (x + \lambda, y).$$

This symmetry does not map to another solution curve, because the point $(\hat{x}, \hat{y})$ is on the same solution curve as (x, y).

1.3 Lie Groups

A *Lie group* is a group of symmetries with a parameter $\lambda \in \mathbb{R}$. Lie group symmetries are functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let A be a set of points $(x, y) \in \mathbb{R}^2$ and let B be a set of points $(\hat{x}, \hat{y}) \in \mathbb{R}^2$. A Lie group P_λ maps A to B:

$$P_\lambda : A \mapsto B.$$

Here, $\hat{x}$ is a function of x, y, and λ and $\hat{y}$ is also a function of x, y, and λ. Therefore, a Lie group can also be written as

$$P_\lambda : (x, y) \mapsto (f(x, y, \lambda), g(x, y, \lambda)).$$

Lie groups must meet the following restrictions:

1) P_λ is one-to-one and onto

2) $P_{\lambda_2} \circ P_{\lambda_1} = P_{\lambda_2+\lambda_1}$

3) $P_0 = I$

4) $\forall \lambda_1 \in \mathbb{R}, \exists \lambda_2 = -\lambda_1$ such that $P_{\lambda_2} \circ P_{\lambda_1} = P_0 = I$

Equation (1.2) furnishes a simple example of a Lie group. The following is another example of a Lie group that is a symmetry for Equation (1.1):

$$P_\lambda : (x, y) \mapsto (\hat{x}, \hat{y}) = (x, e^\lambda y). \tag{1.3}$$

This is a symmetry because it maps a point on one solution curve of Equation (1.1) to a point on another solution curve of Equation (1.1). We will show that it is a symmetry for each fixed λ. It satisfies all of the properties of Lie symmetry groups:

1) The mapping (1.3) is one-to-one and onto.

This means that if two points (x_1, y_1) and (x_2, y_2) map to the same $(\hat{x}, \hat{y})$ then $(x_1, y_1) = (x_2, y_2)$ and for every $(\hat{x}, \hat{y})$ on the interval that (1.3) is defined (in this case, the entirety of $\mathbb{R}^2$), there is a point (x, y) that maps to $(\hat{x}, \hat{y})$. Consider the mapping (1.3) for a fixed λ. We can clearly see that it is both one-to-one and onto for $\hat{x}$ because $\hat{x} = x$. This is the identity mapping. Now consider $\hat{y} = e^{\lambda} y$. If we have $e^{\lambda} y_1 = e^{\lambda} y_2 = \hat{y}$, we can divide by e^{λ} to see that $y_1 = y_2$. To show that there is a y that corresponds with every $\hat{y}$ in $\mathbb{R}$, we can divide $\hat{y}$ by e^{λ}. Then $y = e^{-\lambda}\hat{y}$. Therefore, the mapping (1.3) is one-to-one and onto.

2) The group satisfies the composition property of Lie groups.

Suppose we take $P_{\lambda_2} \circ P_{\lambda_1}$. Apply P_{λ_1} to the point (x, y) to get

$$(\hat{x}_1, \hat{y}_1) = (x, e^{\lambda_1} y).$$

Then apply P_{λ_2} to $(x, e^{\lambda_1} y)$ to get

$$\begin{aligned}(\hat{x}_2, \hat{y}_2) &= (x, e^{\lambda_2} e^{\lambda_1} y)\\ &= (x, e^{\lambda_2 + \lambda_1} y).\end{aligned}$$

This is what we get when we apply $P_{\lambda_2 + \lambda_1}$, and therefore the composition property is satisfied.

3) P_0 is the identity:

$$(\hat{x}, \hat{y}) = (x, e^0 y) = (x, y).$$

4) For all $\lambda_1 \in \mathbb{R}$ there is an inverse. When we apply $P_{-\lambda_1} \circ P_{\lambda_1}$, we get

$$\begin{aligned}(\hat{x}, \hat{y}) &= (x, e^{-\lambda_1}(e^{\lambda_1} y))\\ &= (x, e^{\lambda_1 - \lambda_1} y)\\ &= (x, y).\end{aligned}$$

Therefore, all of the Lie group properties are satisfied, and the mapping (1.3) is a symmetry group for Equation (1.1).

Lie groups might not necessarily be defined over the entire real number plane. We will be dealing with *local groups*. The group action of a *local group* is not necessarily defined over the entire real number plane [5]. Consider the next example.

Example 1.3.1. The following Lie group is only defined if $\lambda < \frac{1}{x}$ when $x > 0$ and $\lambda > \frac{1}{x}$ when $x < 0$:

$$P_\lambda : (x, y) \mapsto (\hat{x}, \hat{y}) = \left(\frac{x}{1 - \lambda x}, \frac{y}{1 - \lambda x}\right). \tag{1.4}$$

We can see at a glance that if $\lambda = \frac{1}{x}$, P_λ is undefined. We can verify that the identity for P_λ is $\lambda = 0$:

$$P_0 : (x, y) \mapsto (\hat{x}, \hat{y}) = \left(\frac{x}{1}, \frac{y}{1}\right) = (x, y).$$

Therefore, the interval on which P_λ is defined must include the origin. If $x > 0$ and $\lambda > \frac{1}{x}$, then the identity is not included in this interval. Similarly, if $x < 0$ and $\lambda < \frac{1}{x}$ then the origin is not included. Therefore, in order for the group to have an identity, it can only be defined when $\lambda < \frac{1}{x}$ for $x > 0$ and $\lambda > \frac{1}{x}$ for $x < 0$. This means that, for a fixed λ, the domain of P_λ is $\frac{1}{-\lambda} < x < \frac{1}{\lambda}$ and $y \in \mathbb{R}$. The range is $\frac{1}{-2\lambda} < \hat{x}$ and $\hat{y} \in \mathbb{R}$.

We can also verify that the Lie group (1.4) satisfies the other three Lie group properties.

1) P_λ is one-to-one and onto. We can show that the mapping (1.4) is one-to-one. If (x_1, y_2) and (x_2, y_2) both map to the same $(\hat{x}, \hat{y})$ then

$$\hat{x} = \frac{x_1}{1 - \lambda x_1} = \frac{x_2}{1 - \lambda x_2}.$$

When we cross multiply we get

$$x_1 - \lambda x_1 x_2 = x_2 - \lambda x_1 x_2.$$

And therefore, $x_1 = x_2$. Similarly,

$$\hat{y} = \frac{y_1}{1 - \lambda x_1} = \frac{y_2}{1 - \lambda x_2}.$$

We have already shown that $x_1 = x_2$ therefore, $y_1 = y_2$. Thus (1.4) is one-to-one.

Now we can show that (1.4) is onto. Consider $\hat{x}$:

$$\hat{x} = \frac{x}{1 - \lambda x}.$$

We can rewrite this equation as:

$$x = (1 - \lambda x)\hat{x}$$

$$x - \lambda x\hat{x} = \hat{x}$$

$$x = \frac{\hat{x}}{1 - \lambda \hat{x}}.$$

Now consider $\hat{y}$:

$$\hat{y} = \frac{y}{1 - \lambda x}.$$

We can rewrite this as

$$\begin{aligned} y &= (1 - \lambda x)\hat{y} = \hat{y} - \lambda x\hat{y} \\ &= \hat{y} - \lambda \hat{y}\left(\frac{\hat{x}}{1 - \lambda \hat{x}}\right) = \frac{\hat{y}}{1 - \lambda \hat{x}}. \end{aligned}$$

Therefore, there is a point (x, y) that corresponds to every point $(\hat{x}, \hat{y})$ so the mapping is onto.

2) The group satisfies the composition property.

Suppose we take $P_{\lambda_2} \circ P_{\lambda_1}$. The domain of this composition is $\frac{1}{-(\lambda_1+\lambda_2)} < x < \frac{1}{(\lambda_1+\lambda_2)}$ and $y \in \mathbb{R}$. The range is $\frac{1}{-2(\lambda_1+\lambda_2)} < \hat{x}$ and $\hat{y} \in \mathbb{R}$. Apply P_{λ_1} to (x, y) to get

$$(\hat{x}_1, \hat{y}_1) = \left(\frac{x}{1-\lambda_1 x}, \frac{y}{1-\lambda_1 x}\right).$$

Then we can apply P_{λ_2} to $(\hat{x}_1, \hat{y}_1)$ to find $(\hat{x}_2, \hat{y}_2)$. For simplicity, we will compute $\hat{x}_2$ first:

$$\begin{aligned}
\hat{x}_2 &= \frac{\hat{x}_1}{1-\lambda_2(\hat{x}_1)} = \frac{\frac{x}{1-\lambda_1 x}}{1-\lambda_2\left(\frac{x}{1-\lambda_1 x}\right)} \\
&= \frac{x}{1-\lambda_1 x} \cdot \frac{1}{\frac{1-\lambda_1 x-\lambda_2 x}{1-\lambda_1 x}} = \frac{x}{1-\lambda_1 x} \cdot \frac{1-\lambda_1 x}{1-\lambda_1 x-\lambda_2 x} \\
&= \frac{x}{1-(\lambda_1+\lambda_2)x}.
\end{aligned}$$

Then we will compute $\hat{y}_2$:

$$\begin{aligned}
\hat{y}_2 &= \frac{\hat{y}_1}{1-\lambda_2(\hat{x}_1)} = \frac{\frac{y}{1-\lambda_1 x}}{1-\lambda_2\left(\frac{x}{1-\lambda_1 x}\right)} \\
&= \frac{y}{1-\lambda_1 x} \cdot \frac{1}{\frac{1-\lambda_1 x-\lambda_2 x}{1-\lambda_1 x}} = \frac{y}{1-\lambda_1 x} \cdot \frac{1-\lambda_1 x}{1-\lambda_1 x-\lambda_2 x} \\
&= \frac{y}{1-(\lambda_1+\lambda_2)x}.
\end{aligned}$$

Therefore, $P_{\lambda_2} \circ P_{\lambda_1}$ yields

$$(\hat{x}_2, \hat{y}_2) = \left(\frac{x}{1-(\lambda_1+\lambda_2)x}, \frac{y}{1-(\lambda_1+\lambda_2)x}\right).$$

Similarly, if we apply $P_{\lambda_1+\lambda_2}$ to (x, y), we get

$$(\hat{x}, \hat{y}) = \left(\frac{x}{1-(\lambda_1+\lambda_2)x}, \frac{y}{1-(\lambda_1+\lambda_2)x}\right).$$

Therefore, the second Lie group property is satisfied.

3) We have already verified that $P_0 = I$, the identity.

4) For all $\lambda_1 \in \mathbb{R}$, there exists an inverse. When we apply $P_{-\lambda_1} \circ P_{\lambda_1}$, we get

$$(\hat{x}, \hat{y}) = \left(\frac{x}{1-(\lambda_1-\lambda_1)x}, \frac{y}{1-(\lambda_1-\lambda_1)x}\right) = (x, y).$$

Therefore, the mapping (1.4) is a Lie group, defined only when $\lambda < \frac{1}{x}$ when $x > 0$ and $\lambda > \frac{1}{x}$ when $x < 0$.

1.4 The Symmetry Condition

The *total derivative operator* is important for understanding the symmetry condition. The total derivative operator is:

$$D_x = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} + \frac{d^2y}{dx^2}\frac{\partial}{\partial y'}.$$

First, consider an example that demonstrates how the total derivative operator works.

Example 1.4.1. We will use the derivative operator to show that the family of ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = c^2 \tag{1.5}$$

satisfies the differential equation

$$y'' = \frac{y'(y - y'x)}{yx}. \tag{1.6}$$

First, apply the total derivative operator to Equation (1.5):

$$\frac{2x}{a^2} + y'\left(\frac{2y}{b^2}\right) = 0.$$

Then divide by 2, and rewrite the equation:

$$\left(\frac{1}{x}\right)\left(\frac{x^2}{a^2}\right) + y'\left(\frac{y}{b^2}\right) = 0.$$

From Equation (1.5), we know that $\frac{x^2}{a^2} = c^2 - \frac{y^2}{b^2}$. When we substitute this in, we get

$$\left(\frac{1}{x}\right)\left(c^2 - \frac{y^2}{b^2}\right) + y'\left(\frac{y}{b^2}\right) = 0.$$

Next, multiply by x

$$c^2 - \frac{y^2}{b^2} + y'\left(\frac{yx}{b^2}\right) = 0.$$

Next, apply the total derivative operator again:

$$\frac{y'y}{b^2} + y'\left(\frac{-2y}{b^2} + \frac{y'x}{b^2}\right) + y''\left(\frac{yx}{b^2}\right) = 0.$$

Then multiply by b^2:

$$y'y + y'(-2y + y'x) + y''(yx) = 0.$$

From here, we can simplify to get

$$y'' = \frac{y'(y - y'x)}{yx}.$$

In general, we are working with differential equations of the form:

$$\frac{dy}{dx} = \omega(x, y). \tag{1.7}$$

In order to satisfy the symmetry condition, the point $(\hat{x}, \hat{y})$ must also be on a solution curve to the differential equation (1.7):

$$\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y}). \tag{1.8}$$

Written with the derivative operator, this looks like:

$$\begin{aligned}\frac{d\hat{y}}{d\hat{x}} &= \frac{D_x\hat{y}}{D_x\hat{x}} \\ &= \frac{\hat{y}_x + \frac{dy}{dx}\hat{y}_y}{\hat{x}_x + \frac{dy}{dx}\hat{x}_y} = \omega(\hat{x}, \hat{y}).\end{aligned}$$

From Equation (1.7), we know that $\frac{dy}{dx} = \omega(x, y)$. Therefore:

$$\frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}). \tag{1.9}$$

Example 1.4.2. We will show that the differential equation

$$\frac{dy}{dx} = \frac{1 - y^2}{x} \tag{1.10}$$

has a symmetry

$$(\hat{x}, \hat{y}) = (e^{\lambda}x, y). \tag{1.11}$$

Substituting this into the symmetry condition, we want to get:

$$\frac{\hat{y}_x + \frac{dy}{dx}\hat{y}_y}{\hat{x}_x + \frac{dy}{dx}\hat{x}_y} = \frac{1 - \hat{y}^2}{\hat{x}}$$

or

$$\frac{\hat{y}_x + \frac{1-y^2}{x}\hat{y}_y}{\hat{x}_x + \frac{1-y^2}{x}\hat{x}_y} = \frac{1 - \hat{y}^2}{\hat{x}}. \tag{1.12}$$

On the right side of Equation (1.12), we get:

$$\frac{1-\hat{y}^2}{\hat{x}} = \frac{1-y^2}{e^{\lambda}x}.$$

On the left side, because $\hat{y}_x = 0$ and $\hat{x}_y = 0$, we get:

$$\begin{aligned}\frac{\hat{y}_x + \frac{1-y^2}{x}\hat{y}_y}{\hat{x}_x + \frac{1-y^2}{x}\hat{x}_y} &= \frac{\frac{1-y^2}{x}}{e^{\lambda}} \\ &= \frac{1-y^2}{e^{\lambda}x}.\end{aligned}$$

Therefore, the symmetry condition is satisfied and (1.11) is a symmetry of (1.10).

Example 1.4.3. In the following example, we will demonstrate that the symmetry

$$(\hat{x}, \hat{y}) = (x, y + \lambda e^{\int F(x)dx}) \tag{1.13}$$

satisfies the differential equation

$$\frac{dy}{dx} = F(x)y + G(x).$$

We can test that the symmetry (1.13) meets the symmetry condition. First, we must calculate $\hat{y}_x$, $\hat{y}_y$, $\hat{x}_x$, and $\hat{y}_x$:

$$\hat{y}_x = F(x)\lambda e^{\int F(x)dx}$$

$$\hat{y}_y = 1$$

$$\hat{x}_x = 1$$

and

$$\hat{x}_y = 0.$$

Beginning with the left side of equation (1.9) and using the partial derivatives calculated above, we obtain

$$\begin{aligned}\frac{d\hat{y}}{d\hat{x}} &= F(x)\lambda e^{\int F(x)dx} + F(x)y + G(x)\\ &= F(x)\left(y + \lambda e^{\int F(x)dx}\right) + G(x).\end{aligned}$$

On the right side, we get

$$\begin{aligned}\omega(\hat{x}, \hat{y}) &= F(\hat{x})\hat{y} + G(\hat{x})\\ &= F(x)\left(y + \lambda e^{\int F(x)dx}\right) + G(x).\end{aligned}$$

Example 1.4.4. In some cases, it is possible to use Equation (1.9) to determine the symmetries of a differential equation. For example, consider

$$\frac{dy}{dx} = y. \tag{1.14}$$

We want to find a symmetry of this equation that satisfies Equation (1.9). In other words, the symmetry should satisfy the following:

$$\frac{\hat{y}_x + y\hat{y}_y}{\hat{x}_x + y\hat{x}_y} = \hat{y}.$$

In order to find the symmetry, one can solve this equation for $\hat{x}$ and $\hat{y}$. To make this easier, one can make assumptions about $\hat{x}$ and $\hat{y}$. Let's assume that $\hat{x}$ is a function of x and y and $\hat{y}$ maps to y

$$(\hat{x}, \hat{y}) = (\hat{x}(x, y), y).$$

Then Equation (1.9) looks like:

$$\frac{y}{\hat{x}_x + y\hat{x}_y} = y.$$

Multiply by $\hat{x}_x + y\hat{x}_y$ on both sides to get

$$y = y(\hat{x}_x + y\hat{x}_y).$$

As long as $y \neq 0$, then

$$1 = \hat{x}_x + y\hat{x}_y.$$

We want to find a symmetry that will satisfy Equation (1.9). The following symmetry does so:

$$(\hat{x}, \hat{y}) = (x + \lambda, y), \lambda \in \mathbb{R}$$

because $\hat{x}_x = 1$ and $\hat{x}_y = 0$.

1.5 A Change in Coordinates

Oftentimes, a differential equation that is difficult to solve in Cartesian coordinates becomes simpler in a different coordinate system. Any ODE that has a symmetry of the form:

$$(\hat{x}, \hat{y}) = (x, y + \lambda) \tag{1.15}$$

can be reduced to quadrature. This means that the differential equation can be solved by an integrating technique. For all λ in some neighborhood of zero, the symmetry condition reduces to:

$$\omega(x, y) = \omega(x, y + \lambda). \tag{1.16}$$

We will differentiate Equation (1.16) with respect to λ at $\lambda = 0$. On the left side of the equation, we get $\frac{\partial}{\partial\lambda}\omega(x, y) = 0$. On the right side, we can use the chain rule to differentiate $\omega(x, y + \lambda)$ with respect to λ and therefore,

$$0 = \frac{\partial\omega}{\partial x}\frac{\partial x}{\partial\lambda} + \frac{\partial\omega}{\partial y}\frac{\partial y}{\partial\lambda}.$$

From Equation (1.16),

$$\frac{\partial x}{\partial\lambda} = 0$$

and

$$\frac{\partial y}{\partial\lambda} = 1.$$

Therefore, we have:

$$0 = \frac{\partial\omega}{\partial y}.$$

Thus the original ODE is a function of x only and then

$$\frac{dy}{dx} = \omega(x)$$

and

$$y = \int \omega(x)dx + c.$$

While a symmetry of the form in Equation (1.16) does not exist in Cartesian coordinates for all differential equations (that would be nice!), it is possible to find such a symmetry in a different coordinate system.

Example 1.5.1. The following differential equation becomes much simpler to integrate when written in polar coordinates

$$\frac{dy}{dx} = \frac{y^3 + x^2y - x - y}{x^3 + xy^2 - x + y}. \tag{1.17}$$

If we let

$$x = r\cos\theta, y = r\sin\theta \tag{1.18}$$

then

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta dr - r\sin\theta d\theta$$

and

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta dr + r\cos\theta d\theta.$$

We will simplify the right side of Equation (1.17) first, then substitute $\frac{dy}{dx}$ on the left. Substituting with Equations (1.18) to the right side, we get:

$$\frac{dy}{dx} = \frac{r^3\sin^3\theta + r^3\cos^2\theta\sin\theta - r\sin\theta - r\cos\theta}{r^3\cos^3\theta + r^3\cos\theta\sin^2\theta - r\cos\theta + r\sin\theta}.$$

This simplifies to:

$$\begin{aligned}\frac{dy}{dx} &= \frac{r^3\sin\theta(\sin^2\theta + \cos^2\theta) - r(\cos\theta + \sin\theta)}{r^3\cos\theta(\cos^2\theta + \sin^2\theta) + r(\sin\theta - \cos\theta)}\\ &= \frac{r^2\sin\theta - \cos\theta - \sin\theta}{r^2\cos\theta + \sin\theta - \cos\theta}\\ &= \frac{\sin\theta(r^2-1) - \cos\theta}{\cos\theta(r^2-1) + \sin\theta}.\end{aligned} \tag{1.19}$$

Now substitute $\frac{dy}{dx}$ from Equation (1.19) above. The goal is to write the equation in terms of $\frac{dr}{d\theta}$:

$$\frac{dy}{dx} = \frac{\sin\theta dr + r\cos\theta d\theta}{\cos\theta dr - r\sin\theta d\theta} = \frac{\sin\theta(r^2-1) - \cos\theta}{\cos\theta(r^2-1) + \sin\theta}.$$

Now one can cross multiply and solve the equation for $\frac{dr}{d\theta}$:

$$(\cos\theta(r^2-1) + \sin\theta)(\sin\theta dr + r\cos\theta d\theta) = (\sin\theta(r^2-1) - \cos\theta)(\cos\theta dr - r\sin\theta d\theta).$$

When we multiply through, we get

$$\begin{aligned}&\sin\theta\cos\theta(r^2-1)dr + r\cos^2\theta(r^2-1)d\theta + \sin^2\theta dr + r\cos\theta\sin\theta d\theta\\&= \sin\theta\cos\theta(r^2-1)dr - r\sin^2\theta(r^2-1)d\theta - \cos^2\theta dr + r\sin\theta\cos\theta.\end{aligned}$$

The $\sin\theta\cos\theta(r^2-1)dr$ term cancels, as well as the $r\cos\theta\sin\theta d\theta$ term, to get:

$$r\cos^2\theta(r^2-1)d\theta + \sin^2\theta dr = -r\sin^2\theta(r^2-1)d\theta - \cos^2\theta dr.$$

This simplifies further:

$$dr = -r\cos^2\theta(r^2-1)d\theta - r\sin^2\theta(r^2-1)d\theta$$

which implies

$$\frac{dr}{d\theta} = -(r^2-1)r(\cos^2\theta + \sin^2\theta)$$

and therefore:

$$\frac{dr}{d\theta} = r(1-r^2). \tag{1.20}$$

This equation is separable in this coordinate system:

$$\frac{dr}{r(1-r^2)} = d\theta.$$

This requires the method of partial fractions to integrate:

$$\int\frac{1}{r}dr - \frac{1}{2}\int\frac{1}{1+r}dr + \frac{1}{2}\int\frac{1}{1-r}dr = \theta.$$

When we integrate we get:

$$\ln r - \frac{1}{2}\ln(1+r) - \frac{1}{2}\ln(1-r) = \theta.$$

and

$$\ln \frac{r}{\sqrt{1-r^2}} = \theta.$$

Equation (1.20) has the symmetry

$$(\hat{r}, \hat{\theta}) = (r, \theta + \lambda)$$

which can be verified with the symmetry condition:

$$\frac{d\hat{r}}{d\hat{\theta}} = \frac{\hat{r}_\theta + \frac{dr}{d\theta}\hat{r}_r}{\hat{\theta}_\theta + \frac{dr}{d\theta}\hat{\theta}_r} = \hat{r}(1-\hat{r}^2).$$

Figure 1.3 shows some of the solution curves for Equation (1.20). The solution curves are rotational symmetries. When written in polar coordinates, the symmetry for Equation (1.20) indicates that the rotational symmetries are translations in θ.

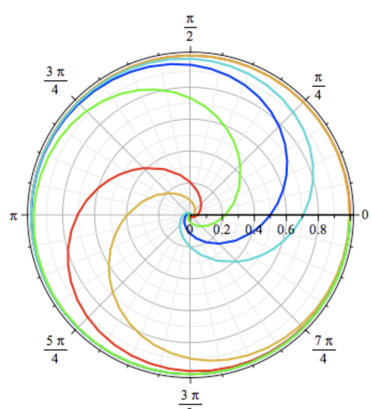

Figure 1.3: Solutions to Equation (1.20)

1.6 Orbits

Orbits are an essential tool for solving differential equations using symmetry methods. Suppose there is a point A on a solution curve to a differential equation. Under a given symmetry, the *orbit* of A is the set of all points that A can be mapped to for all possible values of λ.

Example 1.6.1. Consider the differential equation discussed in Section 1.2:

$$\frac{dy}{dx} = 0.$$

The orbits for the points on the solution curves of this differential equation are vertical lines under the symmetry

$$(\hat{x}, \hat{y}) = (x, y + \lambda).$$

For instance, under the symmetry, the orbit of the point $(1, 0)$ includes

$$\{(1, 1), (1, 2), (1, 3), ...\}.$$

Figure 1.4 shows the orbit of the point $(1, 0)$ under the symmetry $(\hat{x}, \hat{y}) = (x, y + \lambda)$.

Example 1.6.2. We have already seen a symmetry of Equation (1.17) when the equation is expressed in polar coordinates. In Cartesian coordinates, one symmetry of Equation (1.17) is

$$(\hat{x}, \hat{y}) = (x \cos \lambda - y \sin \lambda, x \sin \lambda + y \cos \lambda). \tag{1.21}$$

The orbits of the points on the solution curves of Equation (1.17) are circles. For a

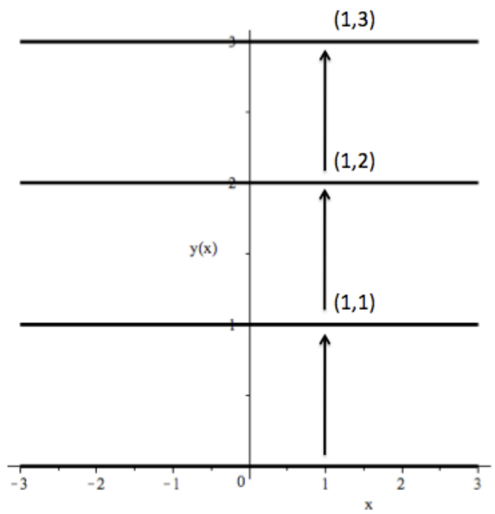

Figure 1.4: Solutions to Equation (1.1) and orbit of the point $(1, 0)$

given point $(x_0, y_0) \neq (0, 0)$, the orbit of the point is

$$r^2 = x_0^2 + y_0^2.$$

To verify this, we will consider $\hat{x}^2 + \hat{y}^2$. Under the symmetry (1.21):

$$\begin{aligned}\hat{x}^2 + \hat{y}^2 &= (x\cos\lambda - y\sin\lambda)^2 + (x\sin\lambda + y\cos\lambda)^2 \\ &= x^2\cos^2\lambda + x^2\sin^2\lambda + y^2\cos^2\lambda + y^2\sin^2\lambda = x^2 + y^2.\end{aligned}$$

We see that $\hat{x}^2 + \hat{y}^2 = x^2 + y^2$. Now we can convert $x^2 + y^2$ to polar coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$ to get

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

and therefore

$$\hat{x}^2 + \hat{y}^2 = x^2 + y^2 = r^2.$$

Thus the new point $(\hat{x}, \hat{y})$ and the original point (x, y) are points on a circle with radius r. We can also see that the orbits of the points on the solution curves of Equation (1.17) are circles by looking at a symmetry of Equation (1.17) in polar coordinates. We saw in Section 1.5 that one symmetry is

$$(\hat{r}, \hat{\theta}) = (r, \theta + \lambda).$$

The radius remains constant as the symmetry takes one solution curve to another.

In Chapter 1 we have seen several examples of Lie symmetry groups and discussed the significance of using a canonical coordinate system to solve first order differential equations. An ODE that has a symmetry of the form

$$(\hat{x}, \hat{y}) = (x, y + \lambda)$$

can be reduced to quadrature. However, a symmetry of this form does not necessarily exist in Cartesian coordinates for a given differential equation, hence the importance of a canonical coordinate system. The next chapter will explain how to find a new coordinate system and how to use it to solve first order ODE's.

CHAPTER 2

Unearthing New Coordinates

The material presented in this chapter is adapted from Chapter Two of *Symmetry Methods for Differential Equations: A Beginner's Guide* [2] and "Solving Differential Equations by Symmetry Groups" [5]. The definitions and examples presented are adapted from these sources. This chapter will explain how to find canonical coordinates and how to use them to solve an ordinary differential equation.

As $\lambda \in \mathbb{R}$ varies under a given symmetry $P_\lambda : (x, y) \mapsto (\hat{x}, \hat{y})$, a point A travels along its orbit. The tangent vectors to an orbit under a given symmetry are crucial to determining the new coordinate system $(r(x, y), s(x, y))$. This chapter explains how to find these tangent vectors, their importance, and how to use them to solve a differential equation by symmetry methods.

2.1 The Tangent Vectors

The tangents to the orbit at any point $(\hat{x}, \hat{y})$ are described by the tangent vector in the x direction, denoted $\xi(\hat{x}, \hat{y})$ and the tangent vector in the y direction, denoted $\eta(\hat{x}, \hat{y})$. Thus

$$\frac{d\hat{x}}{d\lambda} = \xi(\hat{x}, \hat{y})$$

and

$$\frac{d\hat{y}}{d\lambda} = \eta(\hat{x}, \hat{y}).$$

At the initial point (x, y), λ is equal to 0. Therefore

$$\left(\left.\frac{d\hat{x}}{d\lambda}\right|_{\lambda=0}, \left.\frac{d\hat{y}}{d\lambda}\right|_{\lambda=0}\right) = (\xi(x, y), \eta(x, y)).$$

As we will demonstrate in Section 2.2, the tangent vectors $\xi(x, y)$ and $\eta(x, y)$ can be used to find a simplifying coordinate system. However, $\xi(x, y)$ and $\eta(x, y)$ can sometimes be used to find solution curves without the use of different coordinates. The tangent vectors are useful for finding invariant solution curves. An *invariant solution curve* is always mapped to itself under a symmetry. The points on an invariant solution curve are mapped either to themselves or to another point on the same curve [2]. Therefore, the orbit of a noninvariant point on an invariant solution curve is the solution curve itself. If a solution curve is invariant that means that the derivative at the point (x, y) will point in the same direction as the tangent vectors to the orbit [2]. As λ varies, the point is mapped to another point on the same solution curve, rather than a different solution curve. Therefore

$$\frac{dy}{dx} = \omega(x, y) = \frac{\eta(x, y)}{\xi(x, y)}.$$

The *characteristic*, Q is defined by Hydon [2] as

$$Q(x, y, y') = \eta(x, y) - y'\xi(x, y). \tag{2.1}$$

Because $\frac{dy}{dx} = \omega(x, y)$, we can rewrite Equation (2.1) as the *reduced characteristic* $\bar{Q}$:

$$\bar{Q}(x, y) = \eta(x, y) - \omega(x, y)\xi(x, y). \tag{2.2}$$

If, under a given symmetry, the reduced characteristic equals 0, then the solution curve is invariant under that symmetry.

Example 2.1.1. The symmetry

$$(\hat{x}, \hat{y}) = (e^{\lambda} x, e^{(e^{\lambda}-1)x} y) \tag{2.3}$$

acts trivially on

$$\frac{dy}{dx} = y. \tag{2.4}$$

In other words, every solution curve is invariant under the symmetry (2.3). To show this, we will use the reduced characteristic, Equation (2.2). First we must find $\xi(x, y)$ and $\eta(x, y)$. To get $\xi(\hat{x}, \hat{y})$, take the derivative of $\hat{x}$ with respect to λ:

$$\xi(\hat{x}, \hat{y}) = e^{\lambda} x.$$

To find $\xi(x, y)$, evaluate $\xi(\hat{x}, \hat{y})$ at $\lambda = 0$:

$$\xi(x, y) = x.$$

To obtain $\eta(\hat{x}, \hat{y})$ take the derivative of $\hat{y}$ with respect to λ:

$$\eta(\hat{x}, \hat{y}) = e^{\lambda} x e^{(e^{\lambda}-1)x} y.$$

To find $\eta(x, y)$, evaluate $\eta(\hat{x}, \hat{y})$ at $\lambda = 0$:

$$\eta(x, y) = xy.$$

Substituting these into the reduced characteristic, $\bar{Q}$, we get

$$xy - xy = 0.$$

Therefore, the symmetry (2.3) acts trivially on Equation (2.4) because the reduced characteristic equals 0.

Example 2.1.2. Consider the *Riccati equation*:

$$\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3}, x \neq 0. \tag{2.5}$$

It has the following symmetry:

$$(\hat{x}, \hat{y}) = (e^{\lambda}x, e^{-2\lambda}y). \tag{2.6}$$

The tangent vectors are

$$\xi(x, y) = x$$

and

$$\eta(x, y) = -2y.$$

The reduced characteristic is

$$\begin{aligned}\bar{Q}(x, y) &= -2y - \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)x \\ &= -x^2y^2 + \frac{1}{x^2}. \end{aligned} \tag{2.7}$$

Equation (2.7) is equal to 0 when $y = \pm x^{-2}$. Therefore, the symmetry (2.6) acts nontrivially on all the solution curves of (2.5) except for $y = x^{-2}$ and $y = -x^{-2}$. This example will be revisited in Section 2.3.

2.2 Canonical Coordinates

The goal of changing to a different coordinate system is to make a differential equation easier to solve. As demonstrated in Section 1.3, if a differential equation has a symmetry of the form

$$(\hat{x}, \hat{y}) = (x, y + \lambda),$$

then it can be reduced to quadrature and can be solved by an integrating technique. However, not all differential equations have a symmetry of this form in Cartesian coordinates. Therefore, one can change to a new coordinate system in $(r(x, y), s(x, y))$ to obtain a symmetry:

$$P_\lambda : (r, s) \mapsto (\hat{r}, \hat{s}) = (r, s + \lambda).$$

The tangent vectors at (r, s) when $\lambda = 0$ are

$$\left.\frac{d\hat{r}}{d\lambda}\right|_{\lambda=0} = 0 \tag{2.8}$$

and

$$\left.\frac{d\hat{s}}{d\lambda}\right|_{\lambda=0} = 1. \tag{2.9}$$

Applying the chain rule to Equation (2.8) and Equation (2.9), we get

$$\left.\frac{d\hat{r}}{d\lambda}\right|_{\lambda=0} = \left.\frac{d\hat{r}}{dx}\frac{dx}{d\lambda}\right|_{\lambda=0} + \left.\frac{d\hat{r}}{dy}\frac{dy}{d\lambda}\right|_{\lambda=0} = \frac{dr}{dx}\xi(x, y) + \frac{dr}{dy}\eta(x, y) = 0 \tag{2.10}$$

and

$$\left.\frac{d\hat{s}}{d\lambda}\right|_{\lambda=0} = \left.\frac{d\hat{s}}{dx}\frac{dx}{d\lambda}\right|_{\lambda=0} + \left.\frac{d\hat{s}}{dy}\frac{dy}{d\lambda}\right|_{\lambda=0} = \frac{ds}{dx}\xi(x, y) + \frac{ds}{dy}\eta(x, y) = 1. \tag{2.11}$$

Equations (2.10) and (2.11) can also be written as [5]

$$r_x\xi(x,y) + r_y\eta(x,y) = 0 \tag{2.12}$$

and

$$s_x\xi(x,y) + s_y\eta(x,y) = 1. \tag{2.13}$$

Equations (2.12) and (2.13) can be solved using the *method of characteristics* [5]. The solutions to Equations (2.12) and (2.13), can be represented as surfaces, $r(x,y)$ and $s(x,y)$, respectively. First consider Equation (2.12). This equation satisfies:

$$\langle r_x, r_y, -1\rangle \cdot \langle \xi, \eta, 0\rangle = 0.$$

We know that the gradient of $r(x,y) = z$ is $\langle r_x, r_y, -1\rangle$ and therefore, $\langle r_x, r_y, -1\rangle$ is a normal vector to the surface $r(x,y)$. Since the dot product $\langle r_x, r_y, -1\rangle \cdot \langle \xi, \eta, 0\rangle = 0$, the vector $\langle \xi(x,y), \eta(x,y), 0\rangle$ is orthogonal to the normal vector. Thus $\langle \xi(x,y), \eta(x,y), 0\rangle$ is in the tangent plane to $r(x,y)$.

Consider a parameterized curve, C on the surface $r(x,y)$. Because $\langle \xi(x,y), \eta(x,y), 0\rangle$ is in the tangent plane to the surface $r(x,y)$, $\langle \xi(x(t), y(t)), \eta(x(t), y(t)), 0\rangle$ is tangent to C. Therefore, we can write the *symmetric equations* as

$$\frac{dx}{dt} = \xi, \frac{dy}{dt} = \eta, \frac{dz}{dt} = 0.$$

These can also be written:

$$\frac{dx}{\xi} = \frac{dy}{\eta}.$$

We can go through a similar process to find the symmetric equations for Equation

(2.13):

$$\frac{dx}{dt} = \xi, \frac{dy}{dt} = \eta, \frac{dz}{dt} = 1$$

and therefore

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dz.$$

In this case, we will rename z:

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = ds. \tag{2.14}$$

Now consider the function $\phi(x,y)$, the *first integral* of a differential equation:

$$\frac{dy}{dx} = f(x,y). \tag{2.15}$$

First integrals are nonconstant functions whose value is constant along solution curves of Equation (2.15). Therefore, $\phi(x,y) = c$, where c is a constant. Applying the total derivative operator to $\phi(x,y)$, we get

$$\phi_x + f(x,y)\phi_y = 0, \phi_y \neq 0. \tag{2.16}$$

If we divide Equation (2.10) by $\xi(x,y)$, we get

$$\frac{dr}{dx}\frac{\xi(x,y)}{\xi(x,y)} + \frac{dr}{dy}\frac{\eta(x,y)}{\xi(x,y)} = 0$$

and therefore

$$r_x + \frac{\eta(x,y)}{\xi(x,y)} r_y = 0.$$

Comparing this result to Equation (2.16), we find that $r(x,y)$ is a first integral of

$$\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)}, \xi(x,y) \neq 0. \tag{2.17}$$

Therefore, in order to find r, one can solve Equation (2.17). Because $r(x, y)$ is a first integral of Equation (2.17), $r(x, y)$ equals the constant, c.

To find s, one can use Equation (2.14):

$$ds = \frac{dy}{\eta(x, y)} = \frac{dx}{\xi(x, y)}$$

and therefore

$$s = \int \frac{dy}{\eta(x, y)} = \int \frac{dx}{\xi(x, y)}.$$

There is a special case when $\xi(x, y) = 0$. If $\xi(x, y) = 0$, then $r = x$ and $s = \int \frac{dy}{\eta(x,y)}$.

The next example demonstrates how to find the canonical coordinates $r(x, y)$ and $s(x, y)$ with a given one parameter Lie group.

Example 2.2.1. Consider the following one parameter Lie group:

$$(\hat{x}, \hat{y}) = (e^{\lambda}x, e^{k\lambda}y), k > 0. \tag{2.18}$$

To find the tangent vector $\xi(x, y)$, first take the derivative of $\hat{x}(x, y)$ with respect to λ:

$$\xi(\hat{x}, \hat{y}) = e^{\lambda}x. \tag{2.19}$$

Then evaluate Equation (2.19) at $\lambda = 0$:

$$\xi(x, y) = x.$$

To find the tangent vector $\eta(x, y)$, first take the derivative of $\hat{y}(x, y)$ with respect to λ:

$$\eta(\hat{x}, \hat{y}) = ke^{k\lambda}y. \tag{2.20}$$

Then evaluate Equation (2.20) at $\lambda = 0$:

$$\eta(x, y) = ky.$$

Next, we use $\xi(x, y)$ and $\eta(x, y)$ to find $r(x, y)$. Remember that $r(x, y)$ is the first integral of

$$\frac{dy}{dx} = \frac{\eta(x, y)}{\xi(x, y)} = \frac{ky}{x}.$$

This equation is separable:

$$\int \frac{dy}{y} = k \int \frac{dx}{x}.$$

Integrating this we get

$$\ln y = \ln x^k + c_0$$

which simplifies to

$$y = cx^k.$$

When we solve for c, we get $r(x, y) = \frac{y}{x^k}$.

To find s, use Equation (2.14) to get

$$s = \int \frac{dx}{\xi(x, y)} = \int \frac{dx}{x}.$$

Therefore $s = \ln x$. So the canonical coordinates are

$$(r, s) = \left(\frac{y}{x^k}, \ln x\right).$$

Example 2.2.2. Consider the following one parameter Lie group:

$$(\hat{x}, \hat{y}) = \left(\frac{x}{(1 - \lambda x)}, \frac{y}{(1 - \lambda x)}\right). \tag{2.21}$$

We can start by finding the tangent vectors:

$$\xi(\hat{x}, \hat{y}) = \frac{x^2}{(1-\lambda x)^2} \tag{2.22}$$

and

$$\eta(\hat{x}, \hat{y}) = \frac{xy}{(1-\lambda x)^2}. \tag{2.23}$$

Evaluating Equation (2.22) and (2.23) at $\lambda = 0$ results in:

$$(\xi(x, y), \eta(x, y)) = (x^2, xy).$$

Therefore, we can integrate Equation (2.17) to find $r(x, y)$:

$$\frac{dy}{dx} = \frac{\eta(x, y)}{\xi(x, y)} = \frac{y}{x}.$$

This equation is separable:

$$\int \frac{dy}{y} = \int \frac{dx}{x}.$$

When we integrate, we get:

$$\ln y = \ln x + c_0.$$

Exponentiating and solving for r yields:

$$y = cx, c = r = \frac{y}{x}.$$

To find s:

$$s = \int \frac{dx}{x^2} = \frac{-1}{x}.$$

Therefore, the canonical coordinates are

$$(r, s) = \left(\frac{y}{x}, \frac{-1}{x}\right).$$

The reader can see that the tangent vectors, rather than the symmetries themselves, are used to find the canonical coordinates. This section and Section 2.1 explain how to find the tangent vectors and use them when one knows the symmetry for a differential equation. In practice the symmetry is often unknown. Section 2.4 will explain how to find $\xi(x, y)$ and $\eta(x, y)$ without knowing the symmetry itself. Once the tangent vectors and canonical coordinate system are determined, symmetries can be reconstructed from the canonical coordinates. First, solve the canonical coordinates $r(x, y)$ and $s(x, y)$ for x and y to get

$$x = f(r, s), y = g(r, s).$$

Then, $\hat{x}$ and $\hat{y}$ are

$$\hat{x} = f(\hat{r}, \hat{s}) = f(r(x, y), s(x, y) + \lambda) \tag{2.24}$$

and

$$\hat{y} = g(\hat{r}, \hat{s}) = g(r(x, y), s(x, y) + \lambda). \tag{2.25}$$

Example 2.2.3. We can find the symmetry associated with the following canonical coordinates:

$$(r, s) = \left(\frac{y}{x}, \frac{-1}{x}\right).$$

Solving r and s for x and y, we get

$$x = \frac{-1}{s}, y = \frac{-r}{s}.$$

Therefore

$$\begin{aligned}\hat{x} = \frac{-1}{\hat{s}} &= \frac{-1}{s(x,y)+\lambda}\\ &= \frac{-1}{\frac{-1}{x}+\lambda}\\ &= \frac{x}{1-\lambda x}.\end{aligned}$$

And

$$\begin{aligned}\hat{y} = \frac{-\hat{r}}{\hat{s}} &= \frac{-r(x,y)}{s(x,y)}\\ &= \frac{\frac{-y}{x}}{\frac{-1}{x}+\lambda}\\ &= \frac{y}{1-\lambda x}.\end{aligned}$$

So the symmetry is

$$(\hat{x},\hat{y}) = \left(\frac{x}{1-\lambda x}, \frac{y}{1-\lambda x}\right).$$

2.3 A New Way to Solve Differential Equations

Our goal is to write the differential equation in terms of r and s in order to solve it. Then, we can put the solution back into Cartesian coordinates. To find $\frac{ds}{dr}$, apply the total derivative operator to get

$$\frac{ds}{dr} = \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y}. \tag{2.26}$$

This will result in an equation $\frac{ds}{dr}$ written in terms of x and y. To write it in terms of r and s, solve the coordinates $r(x,y)$ and $s(x,y)$ for x and y, then simplify. From there, solve the equation and put the solution back into Cartesian coordinates.

Example 2.3.1. Again, consider the Riccati Equation:

$$\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3}, x \neq 0. \tag{2.27}$$

It has the following symmetry:

$$(\hat{x}, \hat{y}) = (e^{\lambda} x, e^{-2\lambda} y).$$

This can be verified with the symmetry condition, Equation (1.9). First we will calculate:

$$\hat{y}_x = 0, \hat{y}_y = e^{-2\lambda}$$

and

$$\hat{x}_x = e^{\lambda}, \hat{x}_y = 0.$$

Evaluating the left side of Equation (1.9) we get:

$$\frac{d\hat{y}}{d\hat{x}} = \frac{e^{-2\lambda}(xy^2 - \frac{2y}{x} - \frac{1}{x^3})}{e^{\lambda}} = \frac{1}{e^{3\lambda}}\left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)$$

Evaluating the right side of Equation (1.9), we get:

$$\begin{aligned}\frac{dy}{dx} &= e^{\lambda}x(e^{-2\lambda}y)^2 - \frac{2e^{-2\lambda}y}{e^{\lambda}x} - \frac{1}{(e^{\lambda}x)^3} = e^{-3\lambda}xy^2 - e^{-3\lambda}\left(\frac{2y}{x}\right) - e^{-3\lambda}\left(\frac{1}{x^3}\right)\\ &= \frac{1}{e^{3\lambda}}\left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right).\end{aligned}$$

Thus the symmetry condition is satisfied for the symmetry $(\hat{x}, \hat{y}) = (e^{\lambda}x, e^{-2\lambda}y)$ of the Riccati Equation. The tangent vectors for this symmetry are

$$\xi(\hat{x}, \hat{y}) = e^{\lambda}x, \xi(x, y) = x$$

and

$$\eta(\hat{x},\hat{y}) = -2e^{-2\lambda}y, \eta(x,y) = -2y.$$

We can find r using Equation (2.17):

$$\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)} = \frac{-2y}{x}$$

This equation is separable and we can integrate it:

$$\int \frac{1}{y}dy = -2\int \frac{1}{x}dx$$

and therefore

$$\ln y = -2\ln x + c_0.$$

When we exponentiate we get

$$y = cx^{-2.}$$

Then we can solve for c to find r:

$$r = c = x^2 y.$$

We can find s:

$$ds = \frac{dx}{\xi(x,y)} = \frac{dx}{x}$$

and therefore

$$s = \int \frac{dx}{x} = \ln x.$$

The canonical coordinates are

$$(r,s) = (x^2 y, \ln x).$$

We can write x and y in terms of r and s:

$$x = e^s$$

and

$$y = re^{-2s}.$$

When we substitute the canonical coordinates into Equation (2.26), we get

$$\begin{aligned}\frac{ds}{dr} &= \frac{\frac{1}{x}}{2xy + \frac{dy}{dx}x^2} \\ &= \frac{\frac{1}{x}}{2xy + x^3y^2 - 2xy - \frac{1}{x}} \\ &= \frac{1}{x^4y^2 - 1}.\end{aligned}$$

Then substitute in $x = e^s$ and $y = re^{-2s}$:

$$\begin{aligned}\frac{ds}{dr} &= \frac{1}{e^{4s}r^2e^{-4s} - 1} \\ &= \frac{1}{r^2 - 1}.\end{aligned}$$

Then we can integrate. The equation

$$\frac{ds}{dr} = \frac{1}{r^2 - 1}$$

is separable:

$$ds = \frac{dr}{r^2 - 1}$$

and therefore

$$s = \int \frac{dr}{r^2 - 1}.$$

We can integrate this using partial fractions:

$$
\begin{aligned}
s &= \frac{1}{2}\int \frac{1}{r-1}dr - \frac{1}{2}\int \frac{1}{r+1}dr \\
&= \frac{1}{2}(\ln(r-1) - \ln(r+1)) + k_0 \\
&= \frac{1}{2}\ln\left(\frac{r-1}{r+1}\right) + k_0.
\end{aligned}
$$

Then substitute $r = x^2 y$ and $s = \ln x$:

$$\ln x = \frac{1}{2}\ln\left(\frac{x^2 y - 1}{x^2 y + 1}\right) + k_0.$$

We can exponentiate to get

$$x = k\sqrt{\frac{x^2 y - 1}{x^2 y + 1}}.$$

Then we square both sides:

$$x^2 = k\frac{x^2 y - 1}{x^2 y + 1}.$$

Simplification yields:

$$x^2(x^2 y + 1) = k(x^2 y - 1)$$

$$x^4 y + x^2 = kx^2 y - k$$

$$x^4 y - kx^2 y = -k - x^2.$$

The solution to Equation (2.27) is

$$y = \frac{-k - x^2}{x^4 - kx^2}.$$

Recall from Section 2.1 that there are two invariant solution curves to the Riccati equation under the symmetry $(\hat{x}, \hat{y}) = (e^{\lambda}x, e^{-2\lambda}y)$. They are $y = x^{-2}$ and $y = -x^{-2}$.

We obtain $y = -x^{-2}$ when $k = 0$ and $y = x^{-2}$ comes from

$$\lim_{k \to +\infty} \frac{-k - x^2}{x^4 - kx^2} = x^{-2}.$$

Figure 2.1 shows some solution curves of Equation (2.27), including the invariant curves.

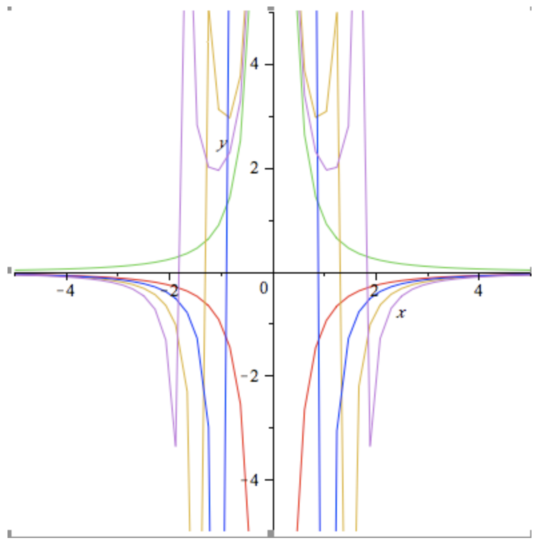

Figure 2.1: Solutions to the Riccati Equation

2.4 The Linearized Symmetry Condition

In all of the previous examples in this IS, the symmetry needed to solve a differential equation has been given. However, in practice it is difficult to find a symmetry that works for a given differential equation. In order to find a symmetry, it is necessary to solve the symmetry condition, Equation (1.9):

$$\frac{\hat{y}_x + \omega(x,y)\hat{y}_y}{\hat{x}_x + \omega(x,y)\hat{x}_y} = \omega(\hat{x}, \hat{y}).$$

This equation gives the symmetry $(x, y) \mapsto (\hat{x}, \hat{y})$. If we could solve this equation for

$\hat{x}$ and $\hat{y}$ then we could find the tangent vectors ξ and η and use them to find new coordinates. However, this equation is often very difficult or impossible to solve. Therefore, it is necessary to use a *linearized symmetry condition* to find $\xi(x, y)$ and $\eta(x, y)$.

We can linearize the symmetry condition using a Taylor series expansion. We can expand $\hat{x}$, $\hat{y}$ and $\omega(\hat{x}, \hat{y})$ around $\lambda = 0$.

$$\hat{x} = x + \lambda\xi(x, y) + O(\lambda^2) \tag{2.28}$$

$$\hat{y} = y + \lambda\eta(x, y) + O(\lambda^2) \tag{2.29}$$

$$\omega(\hat{x}, \hat{y}) = \omega(x, y) + \lambda(\omega_x(x, y)\xi(x, y) + \omega_y(x, y)\eta(x, y)) + O(\lambda^2) \tag{2.30}$$

Here, $O(\lambda^2)$ describes the error function for the Taylor series expansions of $\hat{x}$, $\hat{y}$, and $\omega(\hat{x}, \hat{y})$. Consider an error function for one of the Taylor series expansions above. We will denote it $e(\lambda)$. As λ gets closer to 0, the ratio $\frac{e(\lambda)}{\lambda^2}$ approaches a constant, c. A function $e(\lambda)$ is $O(\lambda^2)$ if:

$$\lim_{\lambda \to 0} \frac{e(\lambda)}{\lambda^2} = c.$$

We ignore terms of $(\lambda)^2$ or higher. For simplicity, $\xi(x, y)$ will be denoted merely as ξ and $\eta(x, y)$ as η.

To obtain the linearized symmetry condition, substitute Equations (2.28), (2.29), (2.30) into the symmetry condition, Equation (1.9). For simplicity purposes, we will compute the numerator, $d\hat{y}$, first, then the denominator, $d\hat{x}$, before simplifying the entire equation. From Equation (2.29), we compute

$$\hat{y}_x = \lambda\eta_x$$

and

$$\hat{y}_y = 1 + \lambda\eta_y.$$

Therefore,

$$\begin{aligned} d\hat{y} &= \lambda\eta_x + \omega(x, y) + \omega(x, y)\lambda\eta_y \\ &= \omega + \lambda(\eta_x + \omega\eta_y) \end{aligned}$$

And from Equation (2.28), we compute

$$\hat{x}_x = 1 + \lambda\xi_x$$

and

$$\hat{x}_y = \lambda\xi_y.$$

Therefore,

$$\begin{aligned} d\hat{x} &= 1 + \lambda\xi_x + \omega(x, y)\lambda\xi_y \\ &= 1 + \lambda(\xi_x + \omega\xi_y). \end{aligned}$$

Now, we obtain that

$$\frac{d\hat{y}}{d\hat{x}} = \frac{\omega + \lambda(\eta_x + \omega\eta_y)}{1 + \lambda(\xi_x + \omega\xi_y)} = \omega(\hat{x}, \hat{y}).$$

Substitute Equation (2.30) for $\omega(\hat{x}, \hat{y})$:

$$\frac{\omega + \lambda(\eta_x + \omega\eta_y)}{1 + \lambda(\xi_x + \omega\xi_y)} = \omega + \lambda(\omega_x\xi + \omega_y\eta).$$

Then we multiply by the denominator to get:

$$\omega + \lambda(\eta_x + \omega\eta_y) = (1 + \lambda(\xi_x + \omega\xi_y))(\omega + \lambda(\omega_x\xi + \omega_y\eta)).$$

Disregarding terms of λ^2 or higher yields:

$$\omega + \lambda(\eta_x + \omega\eta_y) = \omega + \lambda(\omega_x\xi + \omega_y\eta) + \omega\lambda(\xi_x + \omega\xi_y).$$

Simplify further:

$$\eta_x + \omega\eta_y = \omega_x\xi + \omega_y\eta + \omega\xi_x + \omega^2\xi_y.$$

This is the linearized symmetry condition:

$$\eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2 = \xi\omega_x + \eta\omega_y. \tag{2.31}$$

The next three examples demonstrate how to use the linearized symmetry condition.

Example 2.4.1. Consider the equation,

$$\frac{dy}{dx} = \frac{y}{x} + x. \tag{2.32}$$

Substitute Equation (2.32) into the linearized symmetry condition, Equation (2.31) to get

$$\eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2 - (\xi\omega_x + \eta\omega_y) = 0$$
$$\eta_x - \xi_y\left(\frac{y}{x} + x\right)^2 + \left(\eta_y - \xi_x\right)\left(\frac{y}{x} + x\right) - \left(\xi\left(1 - \frac{y}{x^2}\right) + \eta\left(\frac{1}{x}\right)\right) = 0.$$

It is necessary to solve this equation for ξ and η. In its current form, this is a very difficult task. Therefore, we can make an *ansatz* (ansatz!) about ξ and η. Suppose that $\xi = 0$ and η is a function of x only. Then we get

$$\eta_x - \frac{\eta}{x} = 0.$$

This differential equation is easily solved:

$$\int \frac{d\eta}{\eta} = \int \frac{dx}{x}$$

$$\ln \eta = \ln x + c_0$$

$$\eta = cx.$$

Now we can find the canonical coordinates r and s. Recall from Section 2.2 that when $\xi(x, y) = 0$, $r = x$. To find s we can solve

$$ds = \frac{dy}{\eta}.$$

Therefore

$$s = \int \frac{dy}{cx} = \frac{y}{cx}.$$

Now set $c = 1$ to get

$$(r(x, y), s(x, y)) = \left(x, \frac{y}{x}\right)$$

and

$$s_x = \frac{-y}{x^2}, s_y = \frac{1}{x}.$$

Now we can substitute r and s into Equation (2.26) to obtain

$$\begin{aligned} \frac{ds}{dr} &= \frac{\frac{-y}{x^2} + \frac{1}{x}(\frac{y}{x} + x)}{1} \\ &= 1. \end{aligned}$$

Therefore, $s = r + k$, where k is a constant. Substituting x and y back in, we get

$$\frac{y}{x} = x + k.$$

and the general solution to Equation (2.32) is

$$y = x^2 + kx.$$

Example 2.4.2. Consider the differential equation:

$$\frac{dy}{dx} = e^{-x}y^2 + y + e^x. \tag{2.33}$$

We will use the linearized symmetry condition, Equation (2.31), to find the tangent vectors $\xi(x, y)$ and $\eta(x, y)$. First, keep in mind that

$$\omega_x = -e^{-x}y^2 + e^x$$

and

$$\omega_y = 2e^{-x}y + 1.$$

We will make an ansatz about the form of $\xi(x, y)$ and $\eta(x, y)$. Suppose $\xi = 1$ and η is a function of y only. Therefore, Equation (2.31) looks like

$$\eta_y\omega - \xi\omega_x - \eta\omega_y = 0$$

$$\eta_y(e^{-x}y^2 + y + e^x) - (-e^{-x}y^2 + e^x) - \eta(2e^{-x}y + 1) = 0. \tag{2.34}$$

Simplifying Equation (2.34) yields

$$e^{-x}y(\eta_y y + y - 2\eta) + e^x(\eta_y - 1) + \eta_y y - \eta = 0.$$

Therefore, we know that

$$\eta_y y + y - 2\eta = 0, \tag{2.35}$$

$$\eta_y y - \eta = 0, \tag{2.36}$$

and

$$\eta_y - 1 = 0. \tag{2.37}$$

Solving Equation (2.37),

$$\frac{dy}{d\eta} = 1$$

we get $\eta = y$. This is consistent with the Equations (2.35) and (2.36). Now we can find the canonical coordinates. To find r, solve

$$\frac{dy}{dx} = \frac{\eta}{\xi} = y.$$

This is separable. Integrating, we get $\ln y = x + c_0$ which simplifies to

$$y = ce^x$$

and therefore,

$$r = \frac{y}{e^x}.$$

To find s:

$$s = \int dx = x.$$

Therefore, the canonical coordinates are

$$(r, s) = \left(\frac{y}{e^x}, x\right).$$

Substituting the canonical coordinates into Equation (2.26), we get:

$$\begin{aligned}\frac{ds}{dr} &= \frac{1}{-ye^{-x} + e^{-x}(e^{-x}y^2 + y + e^x)} \\ &= \frac{1}{y^2e^{-2x} + 1}.\end{aligned}$$

Therefore:

$$\frac{ds}{dr} = \frac{1}{r^2 + 1}. \tag{2.38}$$

$$s = \int \frac{1}{r^2 + 1} dr = \arctan(r)$$

$$x = \arctan\left(\frac{y}{e^x}\right)$$

and

$$y = \tan(x)e^x.$$

Example 2.4.3. Consider the following equation:

$$\frac{dy}{dx} = \frac{1 - y^2}{xy} + 1. \tag{2.39}$$

Here, we must make an (ansatz!) about the form of ξ and η. We will assume that

$$\xi = \alpha(x) \tag{2.40}$$

and

$$\eta = \beta(x)y + \gamma(x). \tag{2.41}$$

Therefore, we have

$$\xi_x = \alpha', \xi_y = 0$$

and

$$\eta_x = \beta' y + \gamma', \eta_y = \beta.$$

We can substitute these assumptions into the linearized symmetry condition (2.31). First compute ω_x and ω_y :

$$\omega_x = \frac{y^2 - 1}{x^2 y}$$

and

$$\begin{aligned}
\omega_y &= \frac{(-2y)(xy) - x(1-y^2)}{(xy)^2} = \frac{-2xy^2 - x + xy^2}{(x^2y^2)} \\
&= \frac{-x(1+y^2)}{x^2y^2} = \frac{-(1+y^2)}{xy^2}.
\end{aligned}$$

Then substitute Equation (2.40) and Equation (2.41) into Equation (2.31).

$$\beta' y + \gamma' + (\beta - \alpha')\left(\frac{1-y^2}{xy} + 1\right) = \alpha\left(\frac{y^2-1}{x^2y}\right) - (\beta y + \gamma)\left(\frac{1+y^2}{xy^2}\right) \tag{2.42}$$

Expanding and simplifying Equation (2.42) yields the following:

$$\beta' y + \gamma' + \frac{\beta - \alpha'}{xy} - \frac{(\beta - \alpha')y^2}{xy} + \beta - \alpha' = \frac{\alpha y^2}{x^2 y} - \frac{\alpha}{x^2 y} - \frac{(\beta y + \gamma)}{xy^2} - \frac{(\beta y + \gamma)y^2}{xy^2}$$

$$\beta' y + \gamma' + \frac{\beta - \alpha'}{xy} + \frac{(\alpha' - \beta)y}{x} + \beta - \alpha' = \frac{\alpha y}{x^2} - \frac{\alpha}{x^2 y} - \frac{\beta y}{xy^2} - \frac{\gamma}{xy^2} - \frac{\beta y^3}{xy^2} - \frac{\gamma y^2}{xy^2}$$

$$\beta' y + \gamma' + \frac{\beta - \alpha'}{xy} + \frac{(\alpha' - \beta)y}{x} + \beta - \alpha' = \frac{\alpha y}{x^2} - \frac{\alpha}{x^2 y} - \frac{\beta}{xy} - \frac{\beta y}{x} - \frac{\gamma}{xy^2} - \frac{\gamma}{x}.$$

In comparing the powers of y, we get

$$y^{-2} : \gamma = 0$$

$$y^{-1} : \frac{\beta - \alpha'}{x} = \frac{\beta}{x} - \frac{\alpha}{x^2}$$

and

$$y^0 : \beta = \alpha'.$$

This results in the differential equation

$$\frac{\alpha}{x} + \alpha' = 0. \tag{2.43}$$

Equation (2.43) is separable. We get that

$$\int \frac{d\alpha}{\alpha} = -\int \frac{dx}{x}$$

and therefore

$$\ln \alpha = \ln x^{-1} + k_0.$$

Exponentiating, we get

$$\alpha = kx^{-1}.$$

Therefore,

$$\beta = \alpha' = -kx^{-2}.$$

The tangent vectors are

$$(\xi, \eta) = (kx^{-1}, -kx^{-2}y).$$

We can use the tangent vectors ξ and η to find the new coordinates r and s. We know that

$$\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{-kx^{-2}y}{kx^{-1}} = \frac{-y}{x}$$

Integrating, we get

$$\ln y = -\ln x + c_0$$

and

$$y = \frac{c}{x}.$$

Therefore

$$r = c = xy.$$

We can also find s:

$$s = \int \frac{dx}{\xi} = \int x dx = \frac{1}{2}x^2.$$

The new coordinates are

$$(r, s) = \left(xy, \frac{1}{2}x^2\right).$$

Solving these for x and y, we get $x = \sqrt{2s}$ and $y = \frac{r}{\sqrt{2s}}$.

To get the differential equation written in the new coordinates, we will use the total derivative operator:

$$\begin{aligned}\frac{ds}{dr} &= \frac{s_x + \omega s_y}{r_x + \omega r_y} \\ &= \frac{x}{y + (\frac{1-y^2+xy}{xy})x} \\ &= \frac{xy}{1+xy}.\end{aligned}$$

Substituting in $x = \sqrt{2s}$ and $y = \frac{r}{\sqrt{2s}}$ yields the following result:

$$\frac{ds}{dr} = \frac{r}{1+r} = 1 - \frac{1}{r+1}.$$

When we integrate this, we get

$$s = \int 1 dr - \int \frac{1}{1+r} = r - \ln(1+r).$$

Substituting $s = \frac{1}{2}x^2$ and $r = xy$, we find that the solution to Equation (2.39) is

$$\frac{1}{2}x^2 = xy - \ln(1+xy).$$

In this chapter, we have seen several examples illustrating how to find and use a canonical coordinate system to solve first order differential equations. In the next chapter, we will explore another tool, one that is useful for working with higher order differential equations.

CHAPTER 3

Infinitesimal Generators

The material presented in this chapter is adapted from *Differential Equations: Their Solution Using Symmetries* by Hans Stephani [6] and from *Symmetry Methods for Differential Equations: A Beginner's Guide* by Peter Hydon [2].

3.1 Infinitesimal Generator

The method described in Chapters 1 and 2 can be used to solve first order differential equations. Many higher order differential equations can be reduced in order with the use of infinitesimal generators [2]. For a one parameter Lie symmetry group $P_\lambda : (x, y) \mapsto (\hat{x}, \hat{y})$ there exists the tangent vectors $\frac{d\hat{x}}{d\lambda}|_{\lambda=0} = \xi(x, y)$ and $\frac{d\hat{y}}{d\lambda}|_{\lambda=0} = \eta(x, y)$. The infinitesimal generator for the symmetry is the partial differential operator

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}. \tag{3.1}$$

Example 3.1.1. The following symmetry:

$$(\hat{x}, \hat{y}) = (x + \lambda, y + \lambda)$$

has an infinitesimal generator

$$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

because $\xi(x, y) = 1$ and $\eta(x, y) = 1$.

Example 3.1.2. Consider the symmetry:

$$(\hat{x}, \hat{y}) = \left(\frac{x}{1 - \lambda y}, \frac{y}{1 - \lambda y}\right).$$

It has the tangent vectors

$$\xi(x, y) = xy$$

and

$$\eta(x, y) = y^2.$$

The infinitesimal generator for this symmetry is

$$X = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.$$

In most cases, it is not necessary to determine the symmetry based on the infinitesimal generator. However, symmetries can be reconstructed from the infinitesimal generators.

Example 3.1.3. Consider the following infinitesimal generator:

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.2}$$

We can see that the tangent vectors are

$$\xi(x, y) = 1$$

and

$$\eta(x, y) = y.$$

Remember that $\xi(x, y)$ is the derivative of $\hat{x}$ with respect to λ, evaluated at $\lambda = 0$.

Similarly, $\eta(x, y)$ is the derivative of $\hat{y}$ with respect to λ, evaluated at $\lambda = 0$. In this example, $\xi(x, y) = 1$ so we could say that

$$\xi(\hat{x}, \hat{y}) = 1.$$

One equation for $\hat{x}$ that satisfies $\xi(\hat{x}, \hat{y}) = 1$ is

$$\hat{x} = x + \lambda.$$

Because $\eta(x, y) = y$, one possible equation for $\eta(\hat{x}, \hat{y})$ is:

$$\eta(\hat{x}, \hat{y}) = e^{\lambda} y$$

and $\hat{y}$ can be written as

$$\hat{y} = e^{\lambda} y.$$

Therefore, the symmetry corresponding to this infinitesimal generator is

$$(\hat{x}, \hat{y}) = (x + \lambda, e^{\lambda} y).$$

Other possibilities exist for symmetries that satisfy Equation (3.2). For example, $(\hat{x}, \hat{y}) = (2x + \lambda, e^{\lambda} y)$ would also satisfy (3.2).

The next example is more challenging. In the previous example, we can determine the symmetries simply by looking at $\xi(x, y)$ and $\eta(x, y)$. This is not obvious in the following example. One must use $\xi(x, y)$ and $\eta(x, y)$ to find the canonical coordinates. Using the canonical coordinates, one can reconstruct the symmetry by the method described in Section 2.2.

Example 3.1.4. We will reconstruct the symmetry that corresponds to the following infinitesimal generator:

$$X = (1 + x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{3.3}$$

We know that $\xi(x, y) = 1 + x^2$ and $\eta(x, y) = xy$. Now one can find the canonical coordinates. To find $r(x, y)$, solve

$$\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{xy}{1 + x^2}.$$

We get

$$\frac{dy}{y} = \frac{xdx}{1 + x^2}.$$

To integrate, it is necessary to do a u-substitution:

$$u = 1 + x^2$$

and

$$du = 2x.$$

Now we can solve

$$\int \frac{dy}{y} = \frac{1}{2}\int \frac{du}{u}$$

to get

$$\ln y = \frac{1}{2}\ln u + c.$$

This becomes

$$\ln y = \ln \sqrt{1 + x^2} + c_0.$$

When we exponentiate both sides, we get

$$y = c\sqrt{1 + x^2}.$$

And then we can solve for r:

$$r = \frac{y}{\sqrt{1+x^2}}.$$

Now one can determine s using:

$$ds = \frac{dx}{\xi(x,y)},$$

and therefore

$$s = \int \frac{dx}{1+x^2}.$$

Integrating, we get

$$s = \arctan x + c.$$

We can solve $r(x,y)$ and $s(x,y)$ for x and y:

$$x = \tan s$$

and

$$y = r\sqrt{1+\tan^2 s}.$$

Now we use the following equations from Section 2.2, Equations (2.24) and (2.25):

$$\hat{x} = f(\hat{r},\hat{s}) = f(r(x,y), s(x,y)+\lambda)$$

$$\hat{y} = g(\hat{r},\hat{s}) = g(r(x,y), s(x,y)+\lambda).$$

For this generator, Equation (3.3), $\hat{x}$ looks like:

$$\hat{x} = \tan\hat{s} = \tan(s+\lambda) = \tan((\arctan x)+\lambda).$$

Here, it can be shown that the addition formula for $\tan(a+b)$ is (Stewart front cover!):

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}. \tag{3.4}$$

Therefore,

$$\hat{x} = \frac{x + \frac{\sin\lambda}{\cos\lambda}}{1 - x\left(\frac{\sin\lambda}{\cos\lambda}\right)}.$$

We can simplify this fraction to get

$$\hat{x} = \frac{x\cos\lambda + \sin\lambda}{\cos\lambda - x\sin\lambda}.$$

Similarly, we can find $\hat{y}$ using Equation (2.25):

$$\begin{aligned}
\hat{y} &= \hat{r}\sqrt{1 + \tan^2\hat{s}} \\
&= r\sqrt{1 + \tan^2(s + \lambda)} \\
&= \frac{y}{\sqrt{1+x^2}} \cdot \sqrt{1 + \tan^2(\arctan x + \lambda)}.
\end{aligned}$$

Now we will use Equation (3.4) to simplify this fraction:

$$\hat{y} = \frac{y}{\sqrt{1+x^2}} \cdot \sqrt{1 + \left(\frac{x + \tan\lambda}{1 - x\tan\lambda}\right)^2}.$$

When we expand and simplify, we get

$$\begin{aligned}
\hat{y} &= \frac{y}{\sqrt{1+x^2}} \cdot \sqrt{\frac{1 + x^2\tan^2\lambda + x^2 + \tan^2\lambda}{1 - 2x\tan\lambda + x^2\tan^2\lambda}} \\
&= y \cdot \sqrt{\frac{(1+x^2)(1+\tan^2\lambda)}{(1+x^2)(1 - x\tan\lambda)^2}} = \frac{y\sqrt{1+\tan^2\lambda}}{1 - x\tan\lambda} \\
&= \frac{y\sqrt{1 + \frac{\sin^2\lambda}{\cos^2\lambda}}}{1 - x\left(\frac{\sin\lambda}{\cos\lambda}\right)} = \frac{y\sqrt{\frac{1}{\cos^2\lambda}}}{\frac{\cos\lambda - x\sin\lambda}{\cos\lambda}}.
\end{aligned}$$

Therefore

$$\hat{y} = \frac{y}{\cos\lambda - x\sin\lambda}.$$

And the symmetry is

$$(\hat{x}, \hat{y}) = \left(\frac{x\cos\lambda + \sin\lambda}{\cos\lambda - x\sin\lambda}, \frac{y}{\cos\lambda - x\sin\lambda}\right).$$

3.2 Infinitesimal Generator in Canonical Coordinates

The infinitesimal generator can be written in the coordinates $(u(x, y))$ and $(v(x, y))$. Let $F(u, v)$ be an arbitrary smooth function. The infinitesimal generator acts on F:

$$XF(u, v) = XF(u(x, y), v(x, y)).$$

By the chain rule, we get

$$X = \xi(u_x F_u + v_x F_v) + \eta(u_y F_u + v_y F_v).$$

Simplifying this, we get

$$X = F_u(\xi u_x + \eta u_y) + F_v(\xi v_x + \eta v_y).$$

Therefore

$$X = XuF_u + XvF_v.$$

We know that F is an arbitrary function, so the infinitesimal generator in the coordinates $u(x, y)$ and $v(x, y)$ is

$$X = Xu\frac{\partial}{\partial u} + Xv\frac{\partial}{\partial v}. \tag{3.5}$$

Now consider the infinitesimal generator for the canonical coordinates r and s. From Equation (3.5) we get

$$X = (Xr)\frac{\partial}{\partial r} + (Xs)\frac{\partial}{\partial s} \tag{3.6}$$

and

$$Xr = \xi(x, y)\frac{dr}{dx} + \eta(x, y)\frac{dr}{dy}.$$

This is Equation (2.10), which we have already dealt with in Chapter 2. Therefore, we know that $Xr = 0$.

Similarly, from Equation (2.11) we get:

$$Xs = \xi(x, y)\frac{ds}{dx} + \eta(x, y)\frac{ds}{dy},$$

which we have already dealt with in Chapter 2. Therefore, we know that $Xs = 1$. Therefore, we know that Equation (3.6) is

$$X = \frac{\partial}{\partial s}$$

in the canonical coordinates $r(x, y)$ and $s(x, y)$.

Infinitesimal generators make it possible to characterize the action of Lie symmetries on functions without using canonical coordinates. This means that they can be extended to equations with more variables. First consider the situation in two variables:

$$F(x, y) = G(r(x, y), s(x, y))$$

and therefore

$$F(\hat{x}, \hat{y}) = G(\hat{r}, \hat{s}) = G(r, s + \lambda).$$

We can expand this using the Taylor Series. At first glance, this seems to be a two variable problem, but we know from Equation (2.10) that $Xr = 0$. The derivative operator on r is 0. We also know that $\hat{r} = r$. Now consider the Taylor Series for $F(\hat{x}, \hat{y}) = G(\hat{r}, \hat{s})$ centered at s:

$$\begin{aligned} F(\hat{x}, \hat{y}) &= G(r, s) + G_s(r, s)(s + \lambda - s) + \frac{G_{ss}(r, s)}{2!}(s + \lambda - s)^2 + \frac{G_{sss}(r, s)}{3!}(s + \lambda - s)^3 + ... \\ &= \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \frac{\partial^j}{\partial s^j} G(r, s). \end{aligned}$$

Because $X = \frac{\partial}{\partial s}$ in the canonical coordinates $r(x, y)$ and $s(x, y)$, we can write:

$$F(\hat{x}, \hat{y}) = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} X^j G(r, s).$$

We know that $G(r, s) = F(x, y)$ so

$$F(\hat{x}, \hat{y}) = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} X^j F(x, y). \tag{3.7}$$

We know that the Taylor series expansion for e^x is

$$e^x = \sum_{j-0}^{\infty} \frac{x^j}{j!}.$$

Therefore, Equation (3.7) can be rewritten as

$$F(\hat{x}, \hat{y}) = e^{\lambda X} F(x, y). \tag{3.8}$$

This notation is useful for studying the symmetries associated with higher order differential equations [2].

The information in this chapter can be expanded to any number of variables. Suppose a function has L variables, $z^1, z^2, ..., z^L$. Then, the Taylor expansion for $\hat{z}^s$ is

$$\hat{z}^s(z^1, z^2, ...z^L; \lambda) = z^s + \lambda \zeta^s(z^1, ...z^L) + O(\lambda^2)$$

where

$$\zeta^s = \frac{d\hat{z}^s}{d\lambda}|_{\lambda=0}.$$

The infinitesimal generator is

$$X = \zeta^s(z^1, z^2, ..., z^L)\frac{\partial}{\partial z^s}.$$

The material presented in this chapter can be used with higher order differential equations. We will see a preliminary example in the next chapter.

CHAPTER 4

Higher Order Differential Equations

This chapter will extend the use of the infinitesimal generator for higher order differential equations. The material presented in this chapter comes from *Symmetry Methods for Differential Equations: A Beginner's Guide* by Peter Hydon [2] and from *Differential Equations: Their Solutions Using Symmetries* by Hans Stephani [6].

4.1 The Symmetry Condition for Higher Order Differential Equations

In this chapter we will consider higher order differential equations of the form

$$y^{(k)} = \omega(x, y, y', ..., y^{(k-1)}). \tag{4.1}$$

Symmetries for higher order differential equations work similarly to symmetries for first order differential equations. The symmetry must map one solution curve to another. The mapping takes x to $\hat{x}$, y to $\hat{y}$, y' to $\hat{y}'$ and so forth, where $\hat{y}^{(k)}$ is defined as

$$\hat{y}^{(k)} = \frac{d^k \hat{y}}{d\hat{x}^k}. \tag{4.2}$$

The k^{th} derivative of $\hat{y}$ is calculated recursively. Remember that the k^{th} derivative of

$\hat{y}$ is the derivative of the $(k-1)^{st}$ derivative of $\hat{y}$. Therefore

$$\hat{y}^k = \frac{d\hat{y}^{(k-1)}}{d\hat{x}} = \frac{D_x\hat{y}^{(k-1)}}{D_x\hat{x}}. \tag{4.3}$$

The symmetry condition for first order differential equations requires that a mapping $P : (x, y) \mapsto (\hat{x}, \hat{y})$ take one solution curve to another. Thus the symmetry condition for a first order differential equation is

$$\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y}). \tag{4.4}$$

It works in the same fashion for higher order differential equations- a mapping takes a solution curve to another solution curve. Therefore, the symmetry condition for higher order differential equations is

$$\frac{d^n\hat{y}}{d\hat{x}^n} = \hat{y}^{(n)} = \omega(\hat{x}, \hat{y}, \hat{y}', ..., \hat{y}^{(n-1)}). \tag{4.5}$$

The following example demonstrates how to verify the symmetry condition, and demonstrates how to compute $\hat{y}^{(k)}$.

Example 4.1.1. Consider the following differential equation:

$$y'' = 0. \tag{4.6}$$

In this example, we will verify that the following symmetry holds for Equation (4.6):

$$(\hat{x}, \hat{y}) = \left(\frac{1}{x}, \frac{y}{x}\right).$$

In other words, we will show that $\hat{y}'' = 0$ under this symmetry. In order to calculate $\hat{y}''$, we must first calculate $\hat{y}'$ using $\hat{y}$ and $\hat{x}$. To begin, we take the partial derivatives of $\hat{y}$ and $\hat{x}$ in order to take the total derivative operator:

$$\hat{y}_x = \frac{-y}{x^2}, \hat{y}_y = \frac{1}{x}$$

and

$$\hat{x}_x = \frac{-1}{x^2}, \hat{x}_y = 0.$$

Then we can use Equation (4.3) to find $\hat{y}'$:

$$\begin{aligned}\hat{y}' &= \frac{d\hat{y}}{d\hat{x}} = \frac{D_x\left(\frac{y}{x}\right)}{D_x\left(\frac{1}{x}\right)} \\ &= \frac{\frac{-y}{x^2} + \frac{y'}{x}}{\frac{-1}{x^2}} = y - y'x.\end{aligned}$$

Then we can use Equation (4.3) and $\hat{y}'$ to find $\hat{y}''$, thus verifying that $\hat{y}''$ is equal to 0:

$$\begin{aligned}\hat{y}'' &= \frac{d\hat{y}'}{d\hat{x}} = \frac{D_x(y - y'x)}{D_x(\frac{1}{x})} \\ &= (-y' + y' + y''(-x))(-x^2) = x^3 y''.\end{aligned}$$

Therefore, the symmetry condition is satisfied because $y'' = 0$. We have just shown that $\hat{y}'' = 0$.

The symmetry $(\hat{x}, \hat{y}) = \left(\frac{1}{x}, \frac{y}{x}\right)$ for Equation (4.6) is actually a discrete symmetry, rather than a continuous symmetry. The general solution to $y'' = 0$ is

$$y = c_1 x + c_2.$$

When one applies the symmetry, the result is

$$\hat{y} = c_1 \hat{x} + c_2,$$

and therefore

$$\frac{y}{x} = \frac{c_1}{x} + c_2.$$

When we multiply by x we get

$$y = c_1 + c_2 x.$$

This symmetry is its own inverse. Applied once, it maps to $y = c_1 + c_2 x$. Applied a second time, it maps back to $y = c_1 x + c_2$.

4.2 The Linearized Symmetry Condition, Revisited

In Section 2.4, we worked with the linearized symmetry condition for first order differential equations. One can use the linearized symmetry condition to determine the tangent vectors $\xi(x, y)$ and $\eta(x, y)$. Similarly, one can use a linearized symmetry condition for higher order differential equations as well. As demonstrated in the next example, the linearized symmetry condition can be used to find the infinitesimal generator for a given symmetry. First we will demonstrate how to find the linearized symmetry condition for higher order differential equations. To begin, here are the Taylor series expansions for $\hat{x}$, $\hat{y}$, and $\hat{y}^{(k)}$:

$$\hat{x} = x + \lambda\xi + O(\lambda^2), \tag{4.7}$$

$$\hat{y} = y + \lambda\eta + O(\lambda^2), \tag{4.8}$$

$$\hat{y}^{(k)} = y^{(k)} + \lambda\left({}^{(k)}\eta\right) + O(\lambda^2). \tag{4.9}$$

Here, ${}^{(k)}\eta$ denotes the tangent vector that corresponds to the k^{th} derivative of $\hat{y}$ [6]:

$${}^{(k)}\eta = \left.\frac{\partial \hat{y}^{(k)}}{\partial \lambda}\right|_{\lambda=0}.$$

If we substitute Equations (4.7), (4.8), and (4.9) into Equation (4.5), we get

$$\hat{y}^{(n)} = \omega(x + \lambda\xi + O(\lambda^2), y + \lambda\eta + O(\lambda^2), y' + \lambda'\eta + O(\lambda^2), ...y^{(n-1)} + \lambda\left({}^{(n-1)}\eta\right) + O(\lambda^2)). \tag{4.10}$$

Using the chain rule to take the derivative of Equation (4.10) with respect to λ (evaluated at $\lambda = 0$), we get the linearized symmetry condition:

$$^{(n)}\eta = \xi\omega_x + \eta\omega_y + \left({}^{(1)}\eta\right)\omega_{y^{(1)}} + ... + \left({}^{(n-1)}\eta\right)\omega_{y^{(n-1)}}. \tag{4.11}$$

We can calculate $^{(n)}\eta$ recursively. Consider, $^{(1)}\eta$. By definition:

$$\hat{y}' = \frac{D_x\hat{y}}{D_x\hat{x}}.$$

First compute $D_x\hat{y}$ using the Taylor series expansion of $\hat{y}$:

$$\begin{aligned} D_x\hat{y} &= \lambda\eta_x + y'(1 + \lambda\eta_y) + O(\lambda^2) \\ &= y' + \lambda\eta_x + \lambda y'\eta_y \\ &= y' + \lambda D_x\eta + O(\lambda^2). \end{aligned}$$

Then compute $D_x\hat{x}$ using the Taylor series expansion of $\hat{x}$:

$$\begin{aligned} D_x\hat{x} &= 1 + \lambda\xi_x + y'(\lambda\xi_y) \\ &= 1 + \lambda D_x\xi + O(\lambda^2). \end{aligned}$$

Therefore

$$\hat{y}' = \frac{D_x\hat{y}}{D_x\hat{x}} = \frac{y' + \lambda D_x\eta + O(\lambda^2)}{1 + \lambda D_x\xi + O(\lambda^2)}. \tag{4.12}$$

Now multiply Equation (4.12) by

$$\frac{1-\lambda D_x\xi}{1-\lambda D_x\xi}$$

and ignore terms of λ^2 or higher, as we did while calculating the linearized symmetry condition for first order differential equations in Section 2.4. We get:

$$\begin{aligned}\hat{y}' &= (y'+\lambda D_x\eta)(1-\lambda D_x\xi)+O(\lambda^2)\\ &= y'+\lambda(D_x\eta - y'D_x\xi)+O(\lambda^2).\end{aligned}$$

When we compare this with Equation (4.9), we find that

$${}^{(1)}\eta = D_x\eta - y'D_x\xi.$$

We can generalize this for ${}^{(k)}\eta$. From Equation (4.3), we know that

$$\hat{y}^{(k)} = \frac{D_x\hat{y}^{(k-1)}}{D_x\hat{x}}.$$

We have already computed $D_x\hat{x}$. Now we will compute $D_x\hat{y}^{(k-1)}$. First remember that

$$\hat{y}^{(k-1)} = y^{(k-1)} + \lambda\left({}^{(k-1)}\eta\right) + O(\lambda^2).$$

Therefore,

$$\begin{aligned}D_x\hat{y}^{(k-1)} &= \lambda\left({}^{(k-1)}\eta_x\right) + y'\lambda\left({}^{(k-1)}\eta_y\right) + y''\lambda\left({}^{(k-1)}\eta_{y'}\right) + \ldots + y^{(k)}\left(1+\lambda\left({}^{(k-1)}\eta_{y^{(k-1)}}\right)\right) + O(\lambda^2)\\ &= y^{(k)} + \lambda\left({}^{(k-1)}\eta_x\right) + y'\lambda\left({}^{(k-1)}\eta_y\right) + y''\lambda\left({}^{(k-1)}\eta_{y'}\right) + \ldots + y^{(k)}\lambda\left({}^{(k-1)}\eta_{y^{(k-1)}}\right) O(\lambda^2)\\ &= y^{(k)} + \lambda D_x\left({}^{(k-1)}\eta\right) + O(\lambda^2).\end{aligned}$$

Thus,

$$\hat{y}^{(k)} = \frac{D_x \hat{y}^{(k-1)}}{D_x \hat{x}} \tag{4.13}$$

$$= \frac{y^{(k)} + \lambda D_x \left({}^{(k-1)}\eta \right) + O(\lambda^2)}{1 + \lambda D_x \xi + O(\lambda^2)}. \tag{4.14}$$

When we multiply Equation (4.13) by

$$\frac{1 - \lambda D_x \xi}{1 - \lambda D_x \xi}$$

we get

$$\begin{aligned}\hat{y}^{(k)} &= y^{(k)} - y^{(k)} \lambda D_x \xi + \lambda D_x^{(k-1)} \eta + O(\lambda^2) \\ &= y^{(k)} + \lambda (D_x^{(k-1)} \eta - y^{(k)} D_x \xi) + O(\lambda^2).\end{aligned}$$

Comparing this result with Equation (4.9), we get

$${}^{(k)}\eta = D_x^{(k-1)} \eta - y^{(k)} D_x \xi. \tag{4.15}$$

As k increases, the number of terms in ${}^{(k)}\eta$ increases exponentially [2]. For this reason, we will only compute ${}^{(1)}\eta$ and ${}^{(2)}\eta$ in this IS. The computation for ${}^{(1)}\eta$ goes as follows:

$$\begin{aligned}{}^{(1)}\eta &= \eta_x + \eta_y (y') - y' (\xi_x + \xi_y y') \\ &= \eta_x + \eta_y y' - y' \xi_x - \xi_y (y')^2 \\ &= \eta_x + y' (\eta_y - \xi_x) - \xi_y (y')^2.\end{aligned}$$

And then we can use ${}^{(1)}\eta$ to compute ${}^{(2)}\eta$:

$$
\begin{aligned}
{}^{(2)}\eta &= D_x^{(1)}\eta - y'' D_x\xi \\
&= \eta_{xx} + y'\eta_{yx} - y'\xi_{xx} - \xi_{yx}(y')^2 + y'(\eta_{xy} + y'\eta_{yy} - y'\xi_{xy} - \xi_{yy}(y')^2) \\
&+ y''(\eta_y - \xi_x - 2\xi_y y') - y''(\xi_x + \xi_y y') \\
&= \eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + (y')^2(\eta_{yy} - 2\xi_{xy}) - \xi_{yy}(y')^3 + (\eta_y - 2\xi_x - 3\xi_y y')y''
\end{aligned}
$$

Substituting ${}^{(2)}\eta$ into Equation (4.11), we find that the the linearized symmetry condition for second order differential equations is

$$
\begin{aligned}
{}^{(2)}\eta &= \eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + (y')^2(\eta_{yy} - 2\xi_{xy}) - \xi_{yy}(y')^3 + (\eta_y - 2\xi_x - 3\xi_y y')y'' \\
&= \xi\omega_x + \eta\omega_y +^{(1)} \eta\omega_{y'}.
\end{aligned}
\tag{4.16}
$$

We will use Equation (4.16) to determine the infinitesimal generator for the following example.

Example 4.2.1. Again, consider the following second order differential equation:

$$y'' = 0. \tag{4.17}$$

Because $y'' = \omega(x, y) = 0$, we know that ${}^{(2)}\eta = 0$. That leaves us with:

$$\eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + (y')^2(\eta_{yy} - 2\xi_{xy}) - \xi_{yy}(y')^3 = 0. \tag{4.18}$$

Equation (4.18) can be split into a system of four equations, called the *determining equations*:

$$\eta_{xx} = 0 \tag{4.19}$$

$$2\eta_{xy} - \xi_{xx} = 0 \tag{4.20}$$

$$\eta_{yy} - 2\xi_{xy} = 0 \tag{4.21}$$

and

$$\xi_{yy} = 0. \tag{4.22}$$

Integrating Equation (4.22) with respect to y twice results in

$$\xi(x, y) = A(x)y + B(x).$$

Then substitute $\xi(x, y)$ into Equation (4.21):

$$\eta_{yy} = 2A'(x). \tag{4.23}$$

When we integrate Equation (4.23) with respect to y, we get

$$\eta_y = 2A'(x)y + C(x).$$

Then integrate with respect to y again to get:

$$\eta(x, y) = A'(x)y^2 + C(x)y + D(x).$$

Now we can substitute $\xi(x, y)$ and $\eta(x, y)$ into Equations (4.20) and (4.19). First, we must compute η_{xy}, ξ_{xx}, and η_{xx}:

$$\eta_{xy} = 2A''(x)y + C'(x)$$

$$\xi_{xx} = A''(x)y + B''(x)$$

and

$$\eta_{xx} = A'''(x)y^2 + C''(x)y + D''(x) = 0. \tag{4.24}$$

Therefore, Equation (4.20) yields:

$$\begin{aligned} 2\eta_{xy} - \xi_{xx} &= 2(2A''(x)y + C'(x)) - (A''(x)y + B''(x)) \\ &= 3A'''(x)y + 2C'(x) - B''(x) = 0. \end{aligned} \tag{4.25}$$

From Equations (4.24) and (4.25), we can determine that $A''(x) = 0$, $C''(x) = 0$, $D''(x) = 0$ and $B''(x) = 2C'(x)$. The general solutions for $A(x)$, $B(x)$, and $C(x)$ are

$$A(x) = c_1x + c_2$$

$$C(x) = c_3x + c_4$$

and

$$D(x) = c_5x + c_6$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are constants. We can solve for $B(x)$:

$$B''(x) = 2C'(x) = 2c_3$$

$$B'(x) = 2c_3x + c_7$$

$$B(x) = c_3x^2 + c_7x + c_8.$$

When we substitute $A(x)$, $B(x)$, $C(x)$, and $D(x)$ back into $\xi(x, y)$ and $\eta(x, y)$, we get:

$$\begin{aligned} \xi(x, y) &= A(x)y + B(x) \\ &= c_1xy + c_2y + c_3x^2 + c_7x + c_8 \end{aligned}$$

and

$$\begin{aligned}\eta(x,y) &= A'(x)y^2 + C(x)y + D(x)\\ &= c_1y^2 + c_3xy + c_4y + c_5x + c_6.\end{aligned}$$

Therefore, the infinitesimal generator for $y'' = 0$ is:

$$X = (c_1xy + c_2y + c_3x^2 + c_7x + c_8)\partial_x + (c_1y^2 + c_3xy + c_4y + c_5x + c_6)\partial_y.$$

In this chapter, we have illustrated how symmetry methods for higher order equations work with one example - the simplest 2nd order differential equation, $y'' = 0$. There are numerous possibilities for using symmetry methods to solve higher order differential equations or to obtain a reduction in order. Several resources for further study of symmetry methods are presented in the conclusion chapter.

CHAPTER 5

Conclusion

This Independent Study has demonstrated just a small portion of the possibilities for using Lie symmetry groups to solve differential equations. Chapters 1 and 2 explained the fundamentals of this method for first order ODEs, while Chapters 3 and 4 described some of the tools necessary to extend the work in Chapters 1 and 2 to higher order differential equations. These tools have applications in a variety of disciplines.

We have only explored a small sampling of the resources available for studying Lie symmetry methods. The main sources for this I.S. offer several more chapters of information pertaining to these methods. Hydon's, *Symmetry Methods for Differential Equations: A Beginner's Guide* explains how to use Lie symmetries with several parameters and also includes methods for solving partial differential equations (PDEs) [2]. Stephani's *Differential Equations: Their Solution Using Symmetries* offers a detailed explanation of symmetry methods for both ODEs and PDEs. Stephani also includes a chapter on solving systems of differential equations [6]. Ibragimov's *Elementary Lie Group Analysis and Ordinary Differential Equations* is another highly relevant resource for anyone wishing to study the applications of Lie symmetry methods and other tools for solving differential equations [3]. This book has a wealth of information on the application of these methods to physics and engineering.

This project proved challenging in a variety of ways. The selection of the topic resulted from a desire to utilize knowledge from multiple mathematics courses at The College of Wooster, particularly Abstract Algebra and Differential Equations. Basic comprehension of the mathematics involved took a significant amount of time and dedication. Starrett's "Solving Differential Equations by Symmetry Groups" provided a basis for understand the fundamental techniques of Lie symmetry methods [5]. His explanations and examples became useful for further study and for learning the first two chapters of Hydon. Much of this I.S. combines these two sources.

The challenge acquired a new dimension in November, when writing commenced. Some difficulty stemmed from communicating the mathematics clearly and effectively while adhering to the constraints of mathematical and scientific writing. In addition, incorporating equations into proper English prose seemed unexpectedly difficult. This resulted in a number of period placement catastrophes in primitive drafts of Chapter 2.

Finishing this project required skills not only from several math classes but also from quite a few other Wooster classes. This included everything from outlining methods learned in First Year Seminar to drafting strategies developed in writing intensive courses to stress management techniques (and increased familiarity with the Greek alphabet!) acquired during study abroad. Overall, the scope of this project has made it a fitting conclusion to my Wooster career.

References

1. R.L. Burden and J.D. Faires, *Numerical analysis*, Brooks/Cole, 2011.

2. Peter E. Hydon, *Symmetry methods for differential equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2000, A beginner's guide. MR 1741548 (2001c:58036)

3. Nail H. Ibragimov, *Elementary Lie group analysis and ordinary differential equations*, Wiley Series in Mathematical Methods in Practice, vol. 4, John Wiley & Sons Ltd., Chichester, 1999. MR 1679646 (2000f:34007)

4. D. Poole, *Linear algebra: A modern introduction*, Available 2011 Titles Enhanced Web Assign Series, Brooks/Cole, 2010.

5. John Starrett, *Solving differential equations by symmetry groups*, Amer. Math. Monthly **114** (2007), no. 9, 778–792. MR 2360906 (2008i:34072)

6. Hans Stephani, *Differential equations*, Cambridge University Press, Cambridge, 1989, Their solution using symmetries. MR 1041800 (91a:58001)

7. J. Stewart, *Calculus*, Stewart's Calculus Series, Brooks/Cole, 2008.

8. D.G. Zill, *A first course in differential equations: With modeling applications*, Brooks/Cole, 2008.