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BRAID COMPUTATIONS FOR THE CROSSING NUMBER OF KLEIN LINKS

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Abstract. Klein links are a non-orientable counterpart to torus knots and links. It is shown that braids representing a subset of Klein links take on the form of a very positive braid after manipulation. Once the braid has reached this form, its number of crossings is the crossing number of the link it represents. Two formulas are proven to calculate the crossing number of $K(m, n)$ Klein links, where $m \geq n \geq 1$. In combination with previous results, these formulas can be used to calculate the crossing number for any Klein link with given values of m and n.

Keywords. 57M25, 57M27

1. Introduction

A key aspect in the classification of distinct knots and links is the crossing number, a link invariant. The crossing number of a link A, denoted $c(A)$, is the minimum number of crossings that can occur in any projection of the link [1]. Through the use of Alexander-Briggs notation, prime links are placed into finite sets based on both their crossing number and number of components [1, 6]. This paper will use Alexander-Briggs notation, specifically corresponding to the labels given by Rolfsen [6], where the 4^2_1 link has four crossings, two components, and is the first link listed with these invariant values. Braid relations are used to simplify the general braid word for Klein links, which allows us to find their minimal number of crossings.

2. Torus links and Klein Links

A torus link is a link that can be placed on the surface of a torus such that it does not cross over itself [1]. Torus links are denoted $T(m, n)$, where m is the number of times the link wraps around the longitude of the torus, and n is the number of times it wraps around the meridian. Torus links are a commonly studied class of links and formulas that can be used to determine many of their invariants are known. Given the values of m and n, the crossing number can be computed with the formula, $c(T(m, n)) = m(n - 1)$ where $m \geq n$ [5, 9].

Similarly, Klein links are links that can be placed on the surface of a once punctured Klein bottle so that they do not intersect themselves. One method used to form this set of Klein links begins with the identified rectangular representation of the Klein bottle seen in Figure 1. For these Klein links, $K(m, n)$, the m strands originating on the left side of the rectangular diagram are placed to remain entirely below the "hole" representing the self-intersection of the once punctured Klein bottle, and the n strands originating from the top remain entirely above the hole [2, 3, 7, 8]. After a link is formed, the Klein bottle is removed and the link is classified based on its invariants.

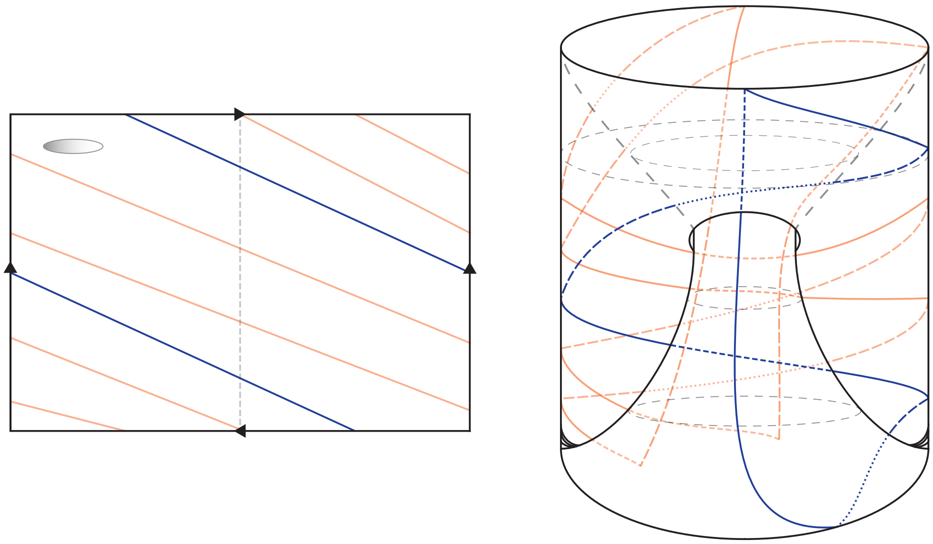

Figure 1. $K(5,3)$ on the identified rectangular representation of a Klein bottle and on the equivalent once punctured Klein bottle. Dashed lines represent portions of the link that lie on hidden surfaces of the Klein bottle.

3. Braids

Braids are a useful technique for representing and classifying links since all links can be represented by braids [1]. A braid is a set of strings connected between a top and bottom bar such that each string always progresses downwards as it crosses above or below the other strings [1, 7]. The strings of an n-braid are numbered from 1 to n, going from the leftmost to the rightmost string. A closed braid representation of a link is formed when these top and bottom bars are connected and the corresponding strings are attached. When describing braids, **braid words** are commonly used due to their simplicity and usefulness. Each crossing is labeled using σ_i^ϵ, where i represents the i^{th} strand of the braid crossing over or under the $(i+1)^{st}$ strand, as illustrated in Figure 2. When the i^{th} strand crosses over the $(i+1)^{st}$ strand $\epsilon = 1$ and when it crosses under $\epsilon = -1$.

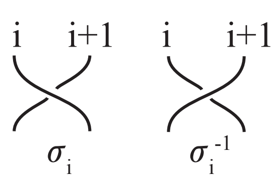

Figure 2. Braid generators [8].

Braids are commonly used to study Klein links and torus links since the corresponding braids are known for given values of m and n. The properties of these braids are exploited to find new properties of the links.

Proposition 1 ([1]). *A general braid word for a torus link is* $(\sigma_1\sigma_2\ldots\sigma_{n-1})^m$ *when* $m \geq 1$ *and* $n \geq 2$.

Proposition 2 ([7, 8]). *A general braid word for a* $K(m,n)$ *Klein link composes the general braid word of a torus link with the half twist,* $\prod_{i=1}^{n-1}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\ldots\sigma_i^{-1})$ *shown in Figure 3, which gives:*

$$K(m,n) = (\sigma_1\sigma_2\ldots\sigma_{n-1})^m \prod_{i=1}^{n-1}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\ldots\sigma_i^{-1}).$$

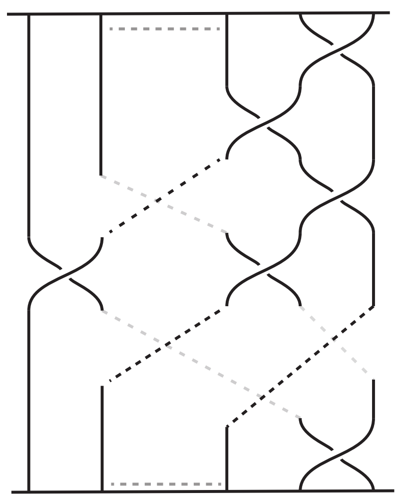

FIGURE 3. A half twist on an n-strand braid [7].

Unlike the general braid word for torus links, the general braid word for Klein links can be manipulated with braid relations to reduce the number of crossings in the braid [5, 9].

Definition 1 ([1, 8]). **Braid relations**, *corresponding to the Reidemeister moves for links, allow a braid to be transformed between equivalent forms without altering the link the closed braid represents. The first three braid moves are shown in Figure 4, conjugation is shown in Figure 5, and stabilization is shown in Figure 6.*

- **Move 1:** $\sigma_i\sigma_i^{-1} = 1 = \sigma_i^{-1}\sigma_i$
- **Move 2:** $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$
- **Move 3:** *For* $|i-j| > 1$, $\sigma_i\sigma_j = \sigma_j\sigma_i$
- **Conjugation:** *For an* n*-string braid word* z, *for* i *from 1 to* $n-1$, $z = \sigma_i z \sigma_i^{-1} = \sigma_i^{-1} z \sigma_i$.
- **Stabilization:** *For an* n*-string braid word* z, $z = z\sigma_n$ *or* $z = z\sigma_n^{-1}$, *resulting in an* $(n+1)$*-string braid word. Also for an* $(n+1)$*-string braid word* z,

assuming z does not contain σ_n or σ_n^{-1}, stabilization allows $z\sigma_n = z$ or $z = z\sigma_n^{-1}$, resulting in an n-string braid word.

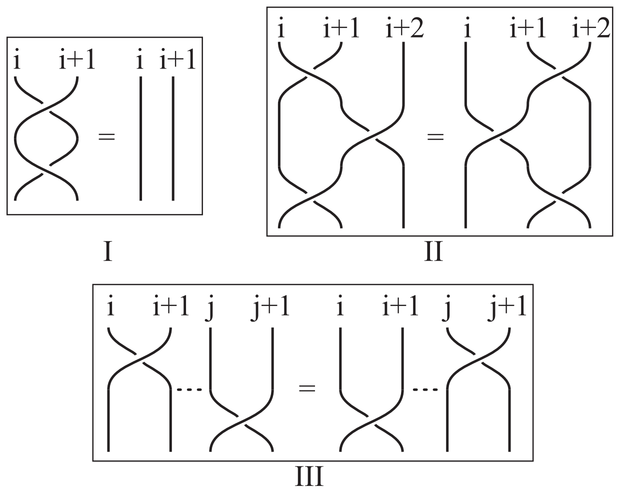

FIGURE 4. Braid moves 1, 2, and 3 [8].

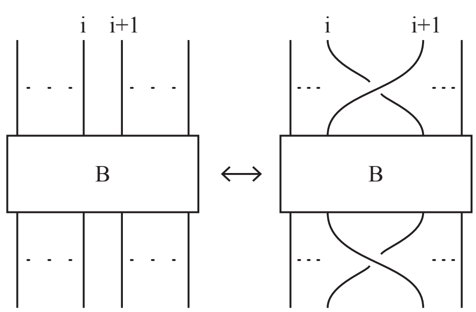

FIGURE 5. Conjugation [8].

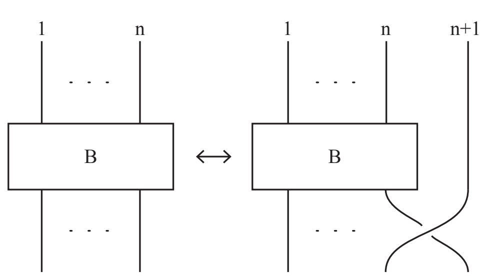

FIGURE 6. Stabilization [8].

When $m \geq n$, a generalized sequence of the first and third braid moves is used to manipulate the general braid word of a Klein link into a form w that untangles the negative half-twist in Lemma 1 below.

Lemma 1. *For $K(m,n)$ where $m \geq n$, a simplified version of the braid word, w, is*

$$w = (\sigma_1\sigma_2 \dots \sigma_{n-1})^{m-n+1}(\sigma_1\sigma_2 \dots \sigma_{n-2})(\sigma_1\sigma_2 \dots \sigma_{n-3}) \dots \sigma_1.$$

Proof. For $m \geq n$, a standard $K(m,n)$ braid can be simplified using the following sequence of braid moves 1 and 3:

$$\begin{aligned}
K(m,n) &= (\sigma_1\sigma_2 \dots \sigma_{n-1})^{m}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1} \dots \sigma_1^{-1})(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1} \dots \sigma_2^{-1}) \dots \sigma_{n-1}^{-1} \\
&= (\sigma_1\sigma_2 \dots \sigma_{n-1})^{m-1}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1} \dots \sigma_2^{-1})(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1} \dots \sigma_3^{-1}) \dots \sigma_{n-1}^{-1} \\
&= (\sigma_1\sigma_2 \dots \sigma_{n-1})^{m-2}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1} \dots \sigma_3^{-1})(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1} \dots \sigma_4^{-1}) \dots \sigma_{n-1}^{-1}\sigma_1 \\
&= (\sigma_1\sigma_2 \dots \sigma_{n-1})^{m-3}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1} \dots \sigma_4^{-1}) \dots \sigma_{n-1}^{-1}\sigma_1\sigma_2\sigma_1 \\
&\vdots \\
&= (\sigma_1\sigma_2 \dots \sigma_{n-1})^{m-n+2}\sigma_{n-1}^{-1}(\sigma_1\sigma_2 \dots \sigma_{n-3})(\sigma_1\sigma_2 \dots \sigma_{n-4}) \dots \sigma_1 \\
&= (\sigma_1\sigma_2 \dots \sigma_{n-1})^{m-n+1}(\sigma_1\sigma_2 \dots \sigma_{n-2})(\sigma_1\sigma_2 \dots \sigma_{n-3}) \dots \sigma_1
\end{aligned}$$

□

In this braid word w, all crossings are positive ($\epsilon = 1$ for all σ_i^{ϵ}), which means it is classified as a homogeneous braid and a positive braid, as defined below.

Definition 2 ([5])**.** *A braid, $\gamma = \sigma_{i_1}^{\epsilon_1} \dots \sigma_{i_k}^{\epsilon_k}$, is a* **homogeneous braid** *if $\epsilon_j = \epsilon_l$ ($\epsilon_i = \pm 1$) whenever $i_j = i_l$.*

Definition 3. *A homogeneous braid a, is a* **positive braid** *if $\epsilon_j = \epsilon_l$ for all σ_i.*

The following definitions and properties provide important information about another class of braids, very positive braids.

Definition 4 ([4])**.** *A braid with r strands has a* **full twist (Δ^2)** *if the braid word contains $(\sigma_1\sigma_2\sigma_3 \dots \sigma_{r-1})^r$.*

Note that a full twist can occur at any point within a braid as shown in Figure 7.

Definition 5 ([4])**.** *A positive braid with a full twist is a* **very positive braid***.*

Definition 6. *The link invariant* **braid index** *is the minimum number of strands needed to represent a link L as a braid, denoted $b(L)$.*

Proposition 3 ([4, 9])**.** *When a braid is a very positive braid p, $b(p) = s$, where s is the number of strands in the very positive braid representation of p.*

Theorem 1 ([5])**.** *A homogeneous n-braid, h, where $b(h) = n$, has the minimal number of crossings for the link it represents.*

These properties are combined to form an important crossing number result for very positive braids.

Lemma 2. *A very positive braid representation of a link has minimal crossings for that link.*

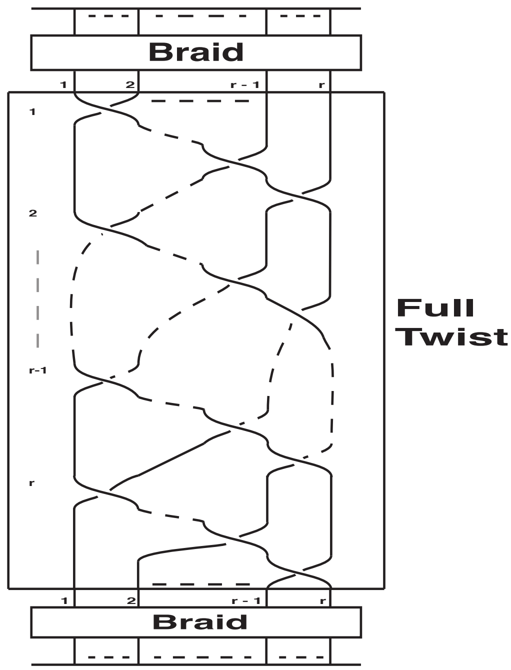

FIGURE 7. A full twist on an r-strand braid.

Proof. Let p be a very positive braid. By Proposition 3 we know $b(p)$ is equal to the number of strands in p and p is a homogeneous braid by Definition 2. Thus, by Theorem 1, a very positive braid contains exactly the number of crossings as the crossing number of the link it represents. □

Very positive braids are useful for determining properties of links since invariants including the crossing number and braid index can be found from braids in this form. For certain values of m and n, w is already in this form and in other cases, the braid word can be simplified into this form. In determining the crossing number for these links, it is useful to know the number of crossings contained within the half-twist of the Klein link braid word.

Lemma 3. *The number of crossings in a half-twist of an n-braid is*

$$\sum_{i=1}^{n-1} i = \frac{n^2 - n}{2}.$$

Proof. The half-twist $\prod_{i=1}^{n-1}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\dots\sigma_i^{-1})$, illustrated in Figure 3, has a crossing for each σ term in the product, or $(n-1)+(n-2)+\dots+2+1$. □

4. Crossing Number Theorem

For certain values of m and n, w is a very positive braid, which means that the crossing number for the corresponding Klein link can be easily determined.

Theorem 2. *For $m \geq n \geq 1$, and $m \geq 2n-1$,*

$$c(K(m,n)) = m(n-1) - \frac{n^2-n}{2}.$$

Proof. Consider the simplified version of the braid word of $K(m,n)$ from Lemma 1:

$$w = (\sigma_1\sigma_2 \dots \sigma_{n-1})^{m-n+1}(\sigma_1\sigma_2 \dots \sigma_{n-2})(\sigma_1\sigma_2 \dots \sigma_{n-3}) \dots \sigma_1.$$

This braid word contains the same number of crossings as $c(T(m,n)) - \frac{n^2-n}{2}$ due to the reduction process in Lemma 1, which removed one crossing from the torus braid for each crossing in the Klein link half-twist corresponding to the use of braid move 1. Referring to Definition 4, this braid word contains a full twist when $m-n+1 \geq n$ since $(\sigma_1\sigma_2\sigma_3 \dots \sigma_{r-1})$ must occur at least r times and $r = n$. Thus, when $m \geq 2n-1$, the simplified braid word will be very positive, and by Lemma 2, will have the minimal number of crossings. □

5. Finding Very Positive Braid Representations

For other values of m and n, a full twist is not contained within w, so only an upper bound on the crossing number is initially known. Since w is a positive braid, stabilization is the only braid relation that can remove crossings. The following example illustrates how braid relations reduce the $K(6,5)$ to a very positive braid. For simplicity, **sub-words** will be specific patterns of consecutive σ_i terms within a braid word.

Example: Let us demonstrate the stabilization process to obtain a full twist on a $K(6,5)$. First we will consider the reduced braid word w of the $K(6,5)$:

$$\underline{\sigma_1\sigma_2\sigma_3}\sigma_4\underline{\sigma_1\sigma_2\sigma_3}\sigma_4\underline{\sigma_1\sigma_2\sigma_3}\sigma_1\sigma_2\sigma_1.$$

One can see that there are two sub-words of $(\sigma_1\sigma_2\sigma_3\sigma_4)$ and three sub-words of $(\sigma_1\sigma_2\sigma_3)$, but these do not satisfy the requirements of a full twist. Thus, when re-examining the braid word, one can see that there are at least three sub-words of $(\sigma_1\sigma_2)$, satisfying the requirements of a full twist if put in order (on a three strand braid):

$$\underline{\sigma_1\sigma_2}\sigma_3\sigma_4\underline{\sigma_1\sigma_2}\sigma_3\sigma_4\underline{\sigma_1\sigma_2}\sigma_3\underline{\sigma_1\sigma_2}\sigma_1.$$

Using braid moves (noted before they are applied), with two stabilizations, we will manipulate the braid word to obtain a full twist:

$K(6,5)$

$= \sigma_1\sigma_2\sigma_3\underline{\sigma_4\sigma_1\sigma_2}\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ (braid move 3)

$= [\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\underline{\sigma_4\sigma_3\sigma_4}\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1]$ (braid move 2, conjugation)

$= \sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1[\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_4\underline{\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1]\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}}$
(braid move 1)

$= \underline{\sigma_3\sigma_1}\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\underline{\sigma_4}$ (braid move 3, **first stabilization**)

$= \sigma_1\underline{\sigma_3\sigma_2\sigma_3}\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\underline{\sigma_3\sigma_1}\sigma_2\sigma_3$ (braid move 2, braid move 1)

$= [\sigma_1\sigma_2\sigma_3\underline{\sigma_2\sigma_1\sigma_2}\sigma_1\sigma_1\sigma_2\sigma_1\underline{\sigma_3\sigma_2\sigma_3}]$ (braid move 2, braid move 2, conjugation)

$= \sigma_3\sigma_2[\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_2\underline{\sigma_3\sigma_2]\sigma_2^{-1}\sigma_3^{-1}}$ (braid move 1)

$= \sigma_3\underline{\sigma_2\sigma_1\sigma_2}\sigma_3\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_2$ (braid move 2)

$= \underline{\sigma_3\sigma_1}\sigma_2\underline{\sigma_1\sigma_3}\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_2$ (braid move 3, braid move 3)

$$= [\sigma_1\underline{\sigma_3\sigma_2\sigma_3}\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_2] \quad \text{(braid move 2, conjugation)}$$
$$= \underline{\sigma_2^{-1}\sigma_1^{-1}[\sigma_1\sigma_2}\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2] \quad \text{(braid move 1)}$$
$$= \underline{\sigma_3}\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 \quad (\textbf{second stabilization})$$
$$= \sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\underline{\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2}.$$

This positive braid contains a full twist after two stabilization moves. Note that this is one way to obtain a full twist, and the full twist may not always appear at the beginning or end of the braid word.

This process of finding the number of stabilization moves needed to find a very positive form of the Klein link is generalized in Theorem 3 below. The set S in Lemma 4, is used to help determine the number of stabilization moves needed to manipulate the braid into a very positive form.

Lemma 4. *The set S, defined as*

$$S = \{k \in \mathbb{Z}^+ |\ \sigma_1\sigma_2\dots\sigma_{k-1} \textit{ occurs at least } k \textit{ times in } w\},$$

is non-empty and finite for $K(m,n)$ when $1 \leq n \leq m < 2n-1$.

Proof. There will always be at least two σ_1 terms in w from Lemma 1, since $m \geq n$ and the exponent $(m-n+1) \geq 1$. Thus, because at least the first term and the last term of the braid word must each be σ_1, $2 \in S$ and S is nonempty. The set S is finite because there are exactly n strands in w; thus if $j > n$, then $j \notin S$. □

Theorem 3. *For $1 \leq n \leq m < 2n-1$,*

$$c(K(m,n)) = m(n-1) - \frac{n^2-n}{2} - \left\lfloor \frac{2n-m}{2} \right\rfloor.$$

Proof. Consider the simplified version of the braid word of $K(m,n)$ from Lemma 1:

$$w = (\sigma_1\sigma_2\dots\sigma_{n-1})^{m-n+1}(\sigma_1\sigma_2\dots\sigma_{n-2})(\sigma_1\sigma_2\dots\sigma_{n-3})\dots\sigma_1.$$

Referring to the definition of a full twist, one can see that this braid word (before manipulation using braid moves) will never contain a full twist because the exponent $m-n+1 < n$. Since there is not a full twist, the braid is positive, but not very positive and the braid index and crossing number remain unknown.

In order to become a very positive braid, a braid representing a Klein link must be transformed so that it is a positive braid with a full twist. Referring to Lemma 4 with $m < 2n-1$, one can identify the presence of at least k sub-words of the form $(\sigma_1\sigma_2\dots\sigma_{k-1})$ where k is a positive integer. Lemma 4 shows S to be non-empty and finite; let $r = max(S)$. Therefore, the sub-word $(\sigma_1\sigma_2\dots\sigma_{r-1})$ occurs at least r times in w.

If a sub-word $(\sigma_1\sigma_2\dots\sigma_{k-1})$ occurs exactly $(k+1)$ times in a braid word w, then r must equal k. This means the sub-word $(\sigma_1\sigma_2\dots\sigma_k)$ must occur k times due to the form of w. Assume $k \neq r$, then $(k+1) \in S$, since $k \neq max(S)$. Since the sub-word $(\sigma_1\sigma_2\dots\sigma_k)$ does not occur $(k+1)$ times, $(k+1) \notin S$; this is a contradiction, and therefore $k = r = max(S)$.

Assume there are $(r+2)$ sub-words of the form $(\sigma_1\sigma_2\dots\sigma_{r-1})$. This implies that there exist $(r+1)$ sub-words of the form $(\sigma_1\sigma_2\dots\sigma_r)$ as seen from the simplified braid word w. This implies that $(r+1) \in S$ and therefore $r \neq max(S)$, which is a contradiction. This means there will not be $(r+2)$ sub-words of the form $(\sigma_1\sigma_2\ldots\sigma_{r-1})$ when $r = max(S)$. Similarly, when there exist more than $(r+2)$ sub-words of the form $(\sigma_1\sigma_2\ldots\sigma_{r-1})$ then there is a value $k \in S$ such that $k > r$ so $r \neq max(S)$, which is a contradiction. Therefore, only r or $(r+1)$ sub-words of the form $(\sigma_1\sigma_2\ldots\sigma_{r-1})$ can exist in the simplified braid word of a Klein link where $m < 2n-1$. We consider these two cases separately.

Case 1. This case examines these simplified braids with r sub-words of the form $(\sigma_1\sigma_2\ldots\sigma_{r-1})$. From the simplified braid word form w, it is known that there are $(m-n+1)$ sub-words of the form $(\sigma_1\sigma_2\ldots\sigma_{n-1})$, where n represents the initial number of strands in the braid. For each stabilization, the number of strands in the braid is decreased by one, and the number of sub-words of $(\sigma_1\sigma_2\ldots\sigma_{n'-1})$, where n' is the number of strands in the braid, is increased by one since the maximum index $(n'-1)$ is decreased with each stabilization. If x is equal to the number of stabilizations that must be used to obtain a full twist, then this relationship gives:

$$(m-n+1)+x = n-x.$$

Solving this equation for x yields

$$x = \frac{2n-m-1}{2}.$$

Case 2. Now this case will examine when $(r+1)$ sub-words of the form $(\sigma_1\sigma_2\ldots\sigma_{r-1})$ are present in the simplified braid word of a Klein link. Similar to Case 1, it is known that there are $(m-n+1)$ sub-words of $(\sigma_1\sigma_2\ldots\sigma_{n-1})$, and each stabilization decreases the number of strands in the braid by one. However, specific to this case, it is known that there is one additional $(\sigma_1\sigma_2\ldots\sigma_{n-1})$ sub-word that is unnecessary in the formation of the full twist. Thus, where x is still the number of stabilizations needed,

$$(m-n+1)-1+x = n-x.$$

Solving this equation for x yields

$$x = \frac{2n-m}{2}.$$

If the two cases are compared, it can be seen that that the values for x only differ by $\frac{1}{2}$. Thus, they can be combined with the following relationship:

$$x = \left\lfloor \frac{2n-m}{2} \right\rfloor.$$

These stabilizations, which reduce the number of strands in the braid, each correspond to the elimination of one crossing from the reduced braid word. Since the resulting braid word contains a full twist and is positive, the braid is very positive, and by Lemma 2, has a minimum number of crossings. Thus,

$$c(K(m,n)) = m(n-1) - \frac{n^2-n}{2} - \left\lfloor \frac{2n-m}{2} \right\rfloor.$$

□

6. Conclusion

These theorems increase our knowledge of Klein links [2, 3, 7, 8], while providing new properties that can be used to find additional connections between torus links and Klein links. With previous results regarding the crossing number for $K(m, n)$ with $m \leq n$ and for $m = 0$ or $n = 0$, the crossing number for any Klein link in this set can be calculated [3, 7]. Through the use of these theorems, we have completed a catalog of Klein links that lists the crossing number, number of components, and complete Alexander-Briggs notation (if available) for all Klein links between $K(1, 0)$ and $K(8, 8)$ [2].

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