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The Planar Rook Algebra

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Abstract

This Independent Study is concerned with examining the diagrammatic algebra known as the planar rook algebra $\mathbb{C}P_n$. Using the tools of representation theory, we decompose $\mathbb{C}P_n$ into the direct sum of P_n-invariant irreducible subspaces. We look to further expand the topic by edge-coloring the diagrams in $\mathbb{C}P_n$. Different results arise when coloring $\mathbb{C}P_n$ with finite abelian versus finite non-abelian groups.

"No no no no no. What?! But no. But yes! Yes yes yes, I see."

- Péter Hermann, at least once per lecture

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Preface

The purpose of this Independent Study is to explore the diagrammatic algebra known as the planar rook algebra.

Notes on Notation

Mathematicians often use different notation to represent the same concept. So, it is necessary for us to address some "ambiguous" notation. First, D_n represents the dihedral group of order $2n$, not of order n. For example, D_4 is the symmetries of the square, not D_8. Second, the term "binary operation" implies closure. Third, we maintain throughout that $0 \notin \mathbb{N}$. Fourth, we will multiply left to right, unless working with the composition of functions (including permutations). Other common notation may be found in the following table.

Notation	Meaning
$H \leq G$	H is a subgroup of G
$H < G$	H is a proper subgroup of G
$H \triangleleft G$	H is a normal subgroup of G (not necessarily proper)
$Z(G)$	the center of G
$C_G(g)$	the centralizer of g in G
$(G \text{ on } \Omega)$	the action of G on Ω
β^G	the orbit of β under the action of G
G_β	the stabilizer of β in G
β^g	the image of β under the action of g
$o(g)$	the order of g
$\mathrm{Tr}(M)$	the trace of a matrix M
$\mathrm{Sym}(\Omega)$	the symmetric group on Ω
$\dim(\mathcal{A})$	the dimension of a vector space or algebra $\mathcal{A}$

CHAPTER 1

Topics in Algebra

We assume knowledge up to the level of completion of introductory courses in group theory and linear algebra. That is, we expect understanding of concepts such as cyclic groups, normal subgroups, homomorphisms, determinant, vector space, etc. To truly understand this Independent Study, however, we need even more background. Many of the examples and theorems in this chapter (especially in Section 1.2) were homework problems from "Advanced Abstract Algebra," a course that ran in the Spring of 2019 through Budapest Semesters in Mathematics.

1.1 Algebraic Structures

Throughout, we will need to call upon a variety of algebraic structures. Algebraic structures are generalizations of systems familiar to most budding mathematicians, such as the integers considered with integer addition and multiplication or continuous functions considered with function addition and composition. Essentially, an algebraic structure consists of a set together with a collection of operations satisfying a list of axioms. Groups and rings are some of the most familiar algebraic structures. A knowledge of groups and vector spaces is assumed, but we will now introduce rings and soon cover other, perhaps more abstract, algebraic structures.

Definition 1.1 Let $(R, +, \cdot)$ be a nonempty set R together with two binary operations $+$ and $\cdot$, which are "addition" and "multiplication," respectively. We say that $(R, +, \cdot)$ is a **ring** if

1. $a + b = b + a$, for all $a, b \in R$.
2. $(a + b) + c = a + (b + c)$, for all $a, b, c \in R$.
3. There exists $0 \in R$ such that $a + 0 = a$, for all $a \in R$.
4. For each $a \in R$, there exists $-a \in R$ such that $a + (-a) = 0$.
5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in R$.
6. For $a, b, c \in R$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

Typically, we write $a \cdot b$ as ab and $(R, +, \cdot)$ as R.

Note that while we require addition to be commutative in R, we do not require multiplication to be commutative. If multiplication is commutative, however, then we say that R is a **commutative ring**. We have an additive identity 0, but lack the requirement of a multiplicative identity. In the case that R has a multiplicative identity 1, we call R a **ring with unity**, a **ring with a** 1 (read "ring with a one"), a **ring with identity**, or simply a **ring with i.d.** In the case that multiplicative inverses exist for the nonzero elements in R, we call R a **division ring**. The multiplicative inverse of $a \in R$ is denoted a^{-1}. Clearly, a division ring is necessarily a ring with i.d. [3].

Considering the first four ring axioms and the binary operation $+$, we see that a ring is really just an abelian group with a second binary operation $\cdot$ that satisfies properties 5 and 6 in Definition 1.1.

Example 1.1. A good first example of a ring is $\mathbb{Z}$ with the usual operations of integer addition and multiplication. This ring is a commutative ring with i.d., but is not a division ring, as it lacks multiplicative inverses.

Example 1.2. Consider $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ with the usual complex addition and multiplication. The ring $\mathbb{Z}[i]$ is called the Gaussian integers and is in fact an algebraic structure more specific than a ring: $\mathbb{Z}[i]$ is an integral domain.

Definition 1.2 Let D be a commutative ring with i.d. Then, D is an **integral domain** if, for every $a, b \in D$ such that $ab = 0$, we have that either $a = 0$ or $b = 0$ [3].

At first glance, this seems standard. In many cases, such as in $\mathbb{R}$, we have that $ab = 0$ implies either $a = 0$ or $b = 0$. Consider, however, $\mathbb{Z}_{15}$, which is not an integral domain (but is a commutative ring with i.d.). In $\mathbb{Z}_{15}$, we have that $3 \cdot 5 = 0$, and neither 3 nor 5 is equal to 0. In this case, 3 and 5 are what are called **zero divisors**, nonzero elements that multiply to be 0. Thus, an equivalent way to think about an integral domain is to consider it as a commutative ring with i.d. that has no zero divisors.

We can essentially combine all of our previous axioms to acquire a field. A **field** is a commutative division ring (and necessarily an integral domain). Common fields include $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_p$, where p is prime. The field $\mathbb{Z}_p$ is often denoted $\mathbb{F}_p$. The **characteristic** of a field is the least positive integer n such that $n \cdot 1 = 0$. In the case that there is no such n, we say that the field in question has characteristic zero.

Definition 1.3 Let R be a ring with multiplicative identity 1. A **left R-module** $\mathcal{M}$ is an abelian group under addition together with a binary operation $\cdot : R \times \mathcal{M} \to \mathcal{M}$ such that for all $r, s \in R$ and $v, w \in \mathcal{M}$, we have that

1. $r \cdot (v + w) = r \cdot v + r \cdot w$,
2. $(r + s) \cdot v = r \cdot v + s \cdot v$,
3. $(rs) \cdot v = r \cdot (s \cdot v)$,
4. $1 \cdot v = v$.

An analogous definition applies to a **right R-module**.

These axioms should look somewhat familiar. A module is really just a "vector space" over a ring with identity. Of course, "vector space" is very intentionally put in quotes here. Vector spaces are, by definition, over fields. However, if we ignore this requirement, we can understand modules. We can think of a vector space as a type of module, as all fields are rings with identity. Furthermore, a vector space is both a left and a right module, as multiplication is commutative in fields. In the case that $\mathcal{M}$ is both a left and right R-module, we simply call $\mathcal{M}$ an R-module.

Example 1.3. Every abelian group G is a $\mathbb{Z}$-module. We know that $\mathbb{Z}$ is a ring with identity 1, and the module axioms hold. To see this, consider our group operation with additive notation. Certainly, for $a, b \in \mathbb{Z}$ and $g, h \in G$,

1. $a \cdot (g + h) = a \cdot g + a \cdot h$,
2. $(a + b) \cdot g = a \cdot g + b \cdot g$,
3. $(ab) \cdot g = a \cdot (b \cdot g)$,
4. $1 \cdot g = v$.

Note that the requirement that G is abelian is very much needed, given that, otherwise, the first module axiom would not necessarily hold.

Definition 1.4 Let $\mathbb{F}$ be a field and V be a vector space over $\mathbb{F}$. Let $\cdot$ be a binary operation from $V \times V$ to V. We say that V is an **algebra** over $\mathbb{F}$ if, for all $\vec{u}, \vec{v}, \vec{w} \in V$ and $\alpha, \beta \in \mathbb{F}$,

1. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$,
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$,
3. $(\alpha\vec{u}) \cdot (\beta\vec{v}) = (\alpha\beta)(\vec{u} \cdot \vec{v})$.

Example 1.4. Take $M_{n\times n}(\mathbb{R})$, the collection of $n \times n$ matrices with entries in $\mathbb{R}$. This is a real vector space, and matrix multiplication satisfies the algebra binary operation in Definition 1.4, as for $A, B, C \in M_{n\times n}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$,

$$(A + B) \cdot C = A \cdot C + B \cdot C,$$

$$A \cdot (B + C) = A \cdot B + A \cdot C,$$

and

$$(\alpha A) \cdot (\beta B) = (\alpha\beta)(A \cdot B).$$

Example 1.5. Take the **group algebra** of a group G over $\mathbb{C}$. This is defined as

$$\mathbb{C}G = \mathbb{C}\text{-span}\left\{\vec{x}_g \mid g \in G\right\},$$

with multiplication defined by $\vec{x}_g\vec{x}_h = \vec{x}_{gh}$. These x_i are abstract vectors indexed by elements of the group G. From linear algebra, we are familiar with the concept of $\mathbb{R}$-span, or just span. Now, we are going beyond real coefficients and using complex coefficients.

Choosing $G = D_3$, for example,

$$\mathbb{C}D_3 = \mathbb{C}\text{-span}\left\{x_e, x_r, x_{r^2}, x_s, x_{sr}, x_{sr^2}\right\}.$$

Elements of $\mathbb{C}D_3$ look like

$$\lambda_1 x_e + \lambda_2 x_r + \lambda_3 x_{r^2} + \lambda_4 x_s + \lambda_5 x_{sr} + \lambda_6 x_{sr^2},$$

where $\lambda_i \in \mathbb{C}$ for all i.

We have seen what happens when we impose axioms on algebraic structures, going, in a sense, beyond groups, but what happens if we start to take away from a group some of its axioms?

Definition 1.5 A **monoid** M is a nonempty set with an associative binary operation that contains an element e such that for all $m \in M$, we have that $em = me = m$.

Unlike elements of groups, elements of monoids are not required to have inverses.

Example 1.6. A good first example of a monoid is $(\mathbb{Z}^+, \cdot)$, where $\mathbb{Z}^+$ is the positive integers and $\cdot$ is integer multiplication. The product of two positive integers is a positive integer, integer multiplication is associative, and 1 is the identity element.

A monoid of particular interest to us is the **rook monoid** R_n, which consists of the set of $n \times n$ matrices with entries in $\mathbb{F}_2$ such that there is at most one 1 in each column and in each row together with matrix multiplication [1]. For example,

$$R_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Of course, the identity matrix I_n is the identity element of R_n, and matrix multiplication is associative. The operation is indeed binary, meaning that R_n is, in fact, a monoid.

The rook monoid is so named because of the positioning of the 1's in the elements of R_n. If we think of $d \in R_n$ as an $n \times n$ chessboard and the 1's as rooks, then d is a chessboard of rooks and empty spaces (0's) in which no two rooks are attacking each other.

Algebraic structures are places in mathematics "where things happen." However, as we will see in the next section, they can also "make things happen."

1.2 Actions

Group actions are fundamental aspects of group theory. Given a group G and a set Ω, the elements of G act on the elements of Ω in that the group elements "do something" to the set elements. Perhaps $g \in G$ maps $\beta_1 \in \Omega$ to $\beta_2 \in \Omega$. The concept of group actions is extremely abstract and will become clearer through example. While we will introduce actions by way of groups, it is important to keep in mind that actions may be performed by other algebraic structures, as well.

Definition 1.6 Let G be a group and Ω a nonempty set. Let $\beta \in \Omega$ and $g_1, g_2 \in G$. Let $\phi : G \times \Omega \to \Omega$ be a map denoted by $\phi(g_1, \beta) := \beta^{g_1}$ such that

1. $\beta^e = \beta$, where e is the identity element of G, and

2. $\beta^{g_1 g_2} = (\beta^{g_1})^{g_2}$.

Then, the map ϕ is a **group action** of G on Ω. We say that G **acts on** Ω and denote the action of G on Ω by (G on Ω).

We will immediately jump into an example in an attempt to clarify, as this definition is more abstract than what is covered in most introductory algebra courses.

Example 1.7. Consider $\Omega = \{1, 2, 3, 4\}$, the set of vertices of a square, and look at

$$D_4 = \{e, (13), (24), (1432), (1234), (12)(34), (14)(23), (12)(24)\}.$$

Each element of D_4 acts on Ω as a function. For example, (1432) acts on 1 by sending it to 4, 2 by sending it to 1, 3 by sending it to 2, and 4 by sending it to 3 (Figure 1.1).

The dihedral groups are special examples of permutation groups. The **symmetric group** on Ω, denoted $\mathrm{Sym}(\Omega)$, is the set of all permutations of the elements of Ω

Figure 1.1: (1432) acting on $\{1, 2, 3, 4\}$

together with composition as the group operation. A **permutation group** is any subgroup of a symmetric group.

A group G acting on a set Ω induces a homomorphism $G \to \mathrm{Sym}(\Omega)$, as each element of G permutes the elements of Ω. This, for example, is justification for Cayley's Theorem.

The object that a group acts on need not be merely a set.

Example 1.8. Let G act on itself via conjugation. That is, for $g, h \in G$, $g^h = h^{-1}gh$. Groups are closed under conjugation by their own elements, and certainly $g^e = ege = g$. For $g, h, k \in G$, we have that

$$g^{hk} = (hk)^{-1}g(hk) = k^{-1}h^{-1}ghk = k^{-1}g^h k = (g^h)^k.$$

Thus, conjugation is indeed a group action.

Other group actions include the **trivial action**, in which $\beta^g = \beta$ for all $g \in G$ and $\beta \in \Omega$, and the **left action** of G on the cosets of a subgroup H of G, in which, for $a \in G$, we set $(aH)^g = gaH$. An analogous definition applies to the **right action** of G on the cosets of H.

Example 1.9. The **left regular action** is one of the easiest group actions to understand, as it is essentially the operation of the acting group. Letting G act on itself via left multiplication, for $g, h \in G$, we have that $g^h = hg$. An analogous definition applies to the **right regular action**.

When a group G acts on a set Ω, it sends elements of Ω to other elements of Ω. Different elements of G often map an element $\beta \in \Omega$ in different ways. Looking at how different elements of G map $\beta \in \Omega$, we encounter the notion of orbits.

Definition 1.7 Let G act on Ω. Let $\beta \in \Omega$. The **orbit** of β under the action of G is the set

$$\beta^G = \{\alpha \in \Omega \mid \text{ there exists } g \in G \text{ such that } \beta^g = \alpha\}.$$

Example 1.10. Let G act on itself via conjugation, as in Example 1.8. Then, the orbit of $g \in G$ under the action is

$$\{h \in G \mid h^{-1}gh = g\}.$$

This set is called the **conjugacy class** of g in G. In general, the orbits of G acting on itself via conjugation are the conjugacy classes of elements in G, or simply, the conjugacy classes of G.

Just as cosets of a subgroup of a group G partition G, orbits of an acted-upon set Ω partition Ω.

Theorem 1.1.

Let G act on Ω. Then, the orbits of the action partition Ω [4].

We will not prove Theorem 1.1, but it is important to keep in mind.

The identity element of an acting group G fixes all elements it acts upon. Other elements of G may behave similarly.

Definition 1.8 Let G act on Ω. The **stabilizer** of an element $\beta \in \Omega$ is the set

$$G_\beta = \{g \in G \mid \beta^g = \beta\}.$$

Example 1.11. Continuing from Example 1.10, where the action was conjugation, we have that for $g \in G$,

$$G_g = C_G(g) = \{h \in G \mid hg = gh\},$$

where $C_G(g)$ is called the **centralizer** of g in G.

Stabilizers are not only subsets of the acting group, but subgroups. This is easy to check and will be not be covered here.

Orbits and stabilizers reveal a great deal of information about their associated action. They allow us to see where elements of the acted-upon set are mapped. They are also related to the order of the acting group.

Theorem 1.2 (Orbit-Stabilizer)**.**
Let G be a group, and let Ω be a finite set. Let G act on Ω, and let $\beta \in \Omega$. Then, $\left|\beta^G\right| = \left|G : G_\beta\right|$.

Proof. Consider $\mathcal{F} : G \to \beta^G$ defined by $\mathcal{F}(g) = \beta^g$. The map is surjective by definition of orbit.

As $G_\beta \leq G$, we have that $\mathcal{F}(g_1) = \mathcal{F}(g_2)$ if and only if $g_1 g_2^{-1} \in G_\beta$. That is, $g_1 \equiv g_2$ mod G_β. Then, there exists a well-defined bijection between $G/G_\beta := \left\{gG_\beta \mid g \in G\right\}$ and β^G. This bijection is given by $gG_\beta \mapsto \beta^g$. Hence, $\left|\beta^G\right| = \left|G/G_\beta\right| = \left|G : G_\beta\right|$, as desired [4]. ◻

We have looked at examples of group actions. We now move to look at types of group actions. There are many types of group actions: ones which have large orbits, ones which partition the acted-upon set in interesting ways, and so on.

Definition 1.9 Let G act on Ω. If, for all $\beta \in \Omega$, we have that $\beta^G = \Omega$, then we say that the action is **transitive** and that G acts **transitively** on Ω.

Example 1.12. The left and right regular actions are transitive.

Note that if the orbit of any single element of Ω is equal to the entirety of Ω, then for every $\beta \in \Omega$, we have that $\beta^G = \Omega$. Thus, Definition 1.9 could be equivalently expressed by changing "for all...we have that" to "there exists...such that."

Definition 1.10 Let G act transitively on Ω. A **block** B is a proper subset of Ω such that $|B| \geq 2$ and given $g \in G$, either $B^g = B$ or $B^g \cap B = \emptyset$.

The set of all blocks is called a system of blocks. A system of blocks forms a partition of Ω.

Definition 1.11 Let G act transitively on Ω. If there exists a system of blocks, we say that G acts **imprimitively** on Ω. Otherwise, we say that G acts **primitively** on Ω.

Example 1.13. Let G act on Ω. If $|\Omega|$ is prime, then G acts primitively.

Proof. As a system of blocks partitions the set being acted upon, and $|\Omega|$ is prime, there are only two ways to partition Ω, either as the set itself or as $|\Omega|$ singletons. Neither of these partitions can form a block system, by definition of a block. Hence, G acts primitively. □

Example 1.14. Consider the action of the dihedral group D_n on the vertices of a regular n-gon. This action is transitive. We claim that D_n acts primitively if and only if n is prime.

Proof. ($\Leftarrow$) By Example 1.13, we have that if n is prime, then D_n acts primitively.

($\Rightarrow$) We proceed by contrapositive. Suppose n is not prime. Let $n = pq$, where neither p nor q is equal to 1. Then, we can partition the vertices of the n-gon into p many sets B_i, where each $|B_i| = q$, and each B_i contains vertices that form a regular q-gon (Figure 1.2). By the way it is defined, the action of the dihedral group preserves this partition of the vertices. That is, the B_i's form a system of blocks, meaning that the action is imprimitive.

This completes the proof. □

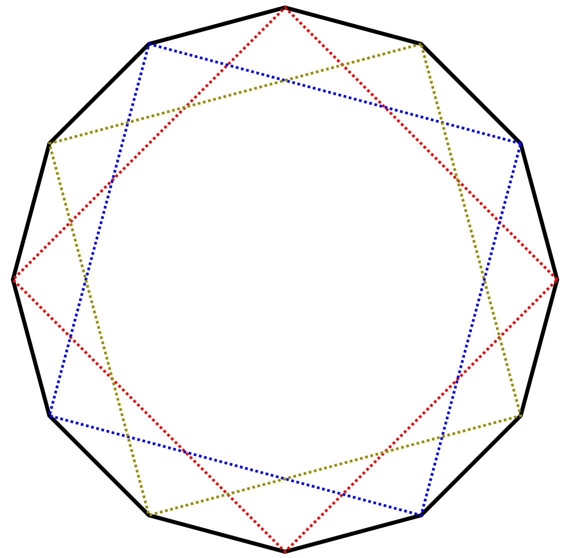


Figure 1.2: A regular 12-gon with its vertices partitioned into 3 regular 4-gons

Transitive and primitive actions affect the stabilizers of the elements of the acted-upon set.

Theorem 1.3.

Let G act transitively on Ω. Then, the stabilizers are conjugates of each other. That is, for G_α and G_β, there exists $g \in G$ such that $G_\alpha = gG_\beta g^{-1}$.

Proof. Let $h \in G_\alpha$. Then, $\alpha^h = \alpha$. As G acts transitively, there exists $g \in G$ such that $\alpha^g = \beta$. Then, $\beta^{g^{-1}} = \alpha$. Thus,

$$\beta^{g^{-1}hg} = \beta^{g^{-1}(hg)} = \alpha^{hg} = (\alpha^h)^g = \alpha^g = \beta.$$

Hence, $g^{-1}hg \in G_\beta$. That is, $h \in gG_\beta g^{-1}$.

The proof of the reverse direction is similar. $\square$

Theorem 1.4.

Let G act transitively on Ω. Then, G acts primitively if and only if $G_\beta \overset{max}{<} G$ (stabilizers are maximal subgroups) [4].

We leave Theorem 1.4 without proof.

Example 1.15. Consider the left regular action of a finite group G on itself. This action is primitive if and only if $|G|$ is prime.

Proof. By Theorem 1.4, we know that the action is primitive if and only if the stabilizer of any element of G is a maximal subgroup of G. In the left regular action, the stabilizer of any element of G is simply $\{e\}$. Then, $\{e\}$ is a maximal subgroup of G if and only if $|G|$ is prime, by Cauchy's Theorem. $\square$

Definition 1.12 Let G act on Ω. Let $g \in G$ such that $\beta^g = \beta$ for all $\beta \in \Omega$. If $g = e$ is the only such g, then we say that G acts **regularly** on Ω.

The previous definition says that a regular action behaves like a function from $\Omega \times \Omega$ to G. Each $(\alpha, \beta) \in \Omega \times \Omega$ has only one element in G to which it corresponds. That is, for any two elements (not necessarily distinct) α and β of Ω, there exists a unique $g \in G$ such that $\alpha^g = \beta$. As $\beta^e = \beta$ for all $\beta \in \Omega$, we can see that the stabilizers of a regular action are trivial. Indeed, this is why the left and right regular actions are named as they are.

We will be interested in the actions of monoids on algebras later in this thesis. Thus, it may be of use to provide an example here.

Example 1.16. Let the monoid $(Z^+, \cdot)$ (Example 1.6) act on the group algebra $\mathbb{C}G$ for some abelian group G (Example 1.5) by

$$\left(\vec{x}_g\right)^z = \vec{x}_{zg},$$

where zg is written in the additive notation. The action of $(\mathbb{Z}^+, \cdot)$ endows $\mathbb{C}G$ with a module-like structure. Thus, we may refer to $\mathbb{C}G$ as a $\mathbb{Z}^+$-module.

1.3 Direct Sums

Direct sums help us understand algebraic structures as the sums of their parts, which simplifies our work in many cases. We will introduce direct sums in the language of vector spaces, but it is important to keep in mind that there are analogous notions for other algebraic structures.

Definition 1.13 Let $\{V_i\}_{i\in I}$ be a collection vector spaces. Define addition in the Cartesian product $\times_{i\in I} V_i$ componentwise and scalar multiplication by $\alpha(\vec{v}, \vec{w}) = (\alpha\vec{v}, \alpha\vec{w})$. Under these operations, $\times_{i\in I} V_i$ is a vector space. Denote $\times_{i\in I} V_i$ defined in this way by

$$\bigoplus_{i\in I} (\text{ex})\ V_i.$$

We call $\bigoplus_{i\in I} (\text{ex})\ V_i$ the **external direct sum** of the V_i's.

Example 1.17. In terms of groups, we can think of an external direct sum as a direct product. That is, the external direct sum of groups G and H is $G \times H$.

Definition 1.14 Let V be a vector space and $\{U_i\}_{i\in I}$ be a collection of subspaces of V such that for all $i \neq j \in I$,

$$U_i \cap U_j = \left\{\vec{0}\right\},$$

and for all $\vec{v} \in V$, there exist $\vec{u}_i \in U_i$ such that

$$\sum_{i\in I} \vec{u}_i = \vec{v}.$$

We may then write V as the sum of the subspaces U_i. Denote V as this sum by

$$V = \bigoplus_{i\in I} (\text{in})\ U_i.$$

We call V the **internal direct sum** of the U_i's.

Example 1.18. Let W be an n-dimensional vector space. Let $\{\vec{u}_1, \ldots, \vec{u}_k\}$ be a basis for a subspace U of W. Then, we can extend this to a basis for W:

$$\{\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots \vec{v}_{n-k}\}.$$

Consider $V = \operatorname{span}\{\vec{v}_1, \ldots \vec{v}_{n-k}\}$. The $\vec{v}_i$'s are linearly independent, so $\{\vec{v}_1, \ldots \vec{v}_{n-k}\}$ is a basis for the subspace V of W. The subspace V has dimension $n-k$, as the basis for W must have n vectors, and the basis for U has k vectors.

We claim that $W = U \oplus_{(\text{in})} V$. To see this, note that $U \cap V = \{\vec{0}\}$, as U and V share no basis vectors. If there were a nonzero vector $\vec{w} \in U \cap V$, then it would be a linear combination of $\vec{u}_i$'s and a linear combination of $\vec{v}_i$'s, which would lead to (without loss of generality) $\vec{u}_i$'s being written as $\vec{v}_i$'s, a contradiction.

Also, $U + V = W$, because every vector in W is a linear combination of $\vec{u}_i$'s and $\vec{v}_i$'s, as seen in the basis for W

$$\{\vec{u_1}, \ldots, \vec{u_k}, \vec{v_1}, \ldots \vec{v_{n-k}}\}.$$

Hence, $W = U \oplus_{(\text{in})} V$.

The internal direct sum is the more appropriate way to think of direct sums as they appear is this thesis. However, will mainly be working with finite direct sums, and in this case, the internal and external direct sums are equivalent. As such, we will use $\bigoplus$ without specifying (in) or (ex).

We are used to working with a vector space in terms of a basis. Not all modules have bases, but those that do are called **free modules**. If $\mathcal{M}$ is a free R-module, then $\mathcal{M}$ is isomorphic to the direct sum of isomorphic copies of R. This, for example, is justification for the fact that finite-dimensional vector spaces of dimensions m and n over a field $\mathbb{F}$ are isomorphic if and only if $m = n$, as they are isomorphic to $\mathbb{F}^m$ and $\mathbb{F}^n$, respectively.

1.4 Representations

Matrix groups are seen throughout algebra, being introduced in a first course in linear algebra and then developed over further study. Given a group, we may be able to represent it as a matrix group and use tools from linear algebra to obtain information. As when introducing actions, we will work in the language of groups, but representation theory is applicable to other algebraic structures, as well.

Example 1.19. Consider $S_3 = \{e, (12), (13), (23), (132), (123)\}$. Here, for example, the element (123) maps 1 to 2, 2 to 3, and 3 to 1. Another way of thinking about the way that an element in S_3 behaves is via matrices. Take not (123), but

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Starting from I_3, we look at how (123) changes the entries:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{(123)} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

In the first row of I_3, the 1 moves to the second position, in the second row, the 1 moves to the third position, and in the third row, the 1 moves to the first position. The matrix on the right is a valid way to represent the element (123) in $\mathrm{GL}_3(\mathbb{F}_2)$. Furthermore, the homomorphism $\varphi : S_3 \to \mathrm{GL}_3(\mathbb{F}_2)$ defined by

$$\text{i.d.} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad (123) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

is a representation of S_3 over $\mathbb{F}_2$. This representation of S_3 is called the **natural representation** and is but one instance of a much broader idea.

Definition 1.15 Let G be a group and V a vector space over a field $\mathbb{F}$. A homomorphism $\varphi : G \to \mathrm{GL}_n(V)$, $n \geq 1$, is called a **representation** of G on V.

Note that $\mathrm{GL}_n(V)$ and $\mathrm{GL}(V^n)$ are closely related. The former is a matrix group, while the latter is a group of bijective linear transformations. The two are isomorphic when a fixed ordered basis for V^n is chosen. We will use language associated with both $\mathrm{GL}_n(V)$ and $\mathrm{GL}(V^n)$, keeping in mind the subtle differences between the groups [5].

The natural representation of S_n is defined analogously to that of S_3 (Example 1.19). Other common representations include the **trivial representation**, in which every element of G maps to the identity matrix, and the **alternating representation** of S_n, in which even and odd σ map to (1) and (-1), respectively. The map of the alternating representation may also be thought of as $\varphi(\sigma) = (-1)^n$, where n is the number of transpositions that make up σ; equivalently, where n is the minimum number of row swaps needed to put the natural representation of σ into reduced row echelon form (in this case, the identity matrix).

Example 1.20. If $\mathcal{A}$ is an algebra, then a representation of $\mathcal{A}$ is an $\mathcal{A}$-module.

From studying linear algebra, we know about linear transformations. If $\mathcal{T}$ is a linear transformation on a vector space V (that is, $\mathcal{T} : V \to V$) and W is a subspace of V, we can look at $\mathcal{T}$-invariance. The subspace W is called **$\mathcal{T}$-invariant** if for all $\overrightarrow{w} \in W$, we have that $\overrightarrow{w}^{\mathcal{T}} \in W$. A similar notion appears in the language of group representations.

Definition 1.16 Let φ be a representation of G on a vector space V. Let W be a subspace of V. Then, W is called G**-invariant** if W is g^{φ}-invariant for all $g \in G$.

Since $g^{\varphi} \in \mathrm{GL}_n(V)$, we have that g^{φ} is a linear transformation from V^n to V^n. This is justification for our use of the term "invariant" here.

Example 1.21. Let φ be the natural representation of S_3 over $\mathbb{F}_2$ (Example 1.19). Consider $W := \{(\alpha, \alpha, \alpha) \mid \alpha \in \mathbb{C}\}$. This is a one-dimensional subspace of the vector space $\mathbb{C}^3$. The subspace W is S_3-invariant, as $(\alpha, \alpha, \alpha)^{g^{\varphi}} = (\alpha, \alpha, \alpha)$ for all g^{φ}, since permutation of the entries of (α, α, α) is irrelevant.

Fix $k \in \mathbb{C}$. The subspace $\{(\alpha, \beta, \gamma) \mid \alpha + \beta + \gamma = k\}$ of $\mathbb{C}^3$ is also S_3-invariant. Elements of the representation of S_3 merely permute α, β, and γ. The sum of the three entries is always k.

Definition 1.17 A representation $\psi : G \to \mathrm{GL}_n(V)$ is called **irreducible** if there do not exist any non-trivial proper G-invariant subspaces of V with respect to ψ.

This definition is not all-encompassing. We will allow for "$\mathrm{GL}_n(V)$" to be replaced with other spaces, such as $\mathbb{C}$. Furthermore, the term "representation" will be used rather generally.

Example 1.22. Any 1-dimensional representation is irreducible, as such a representation cannot contain a non-trivial proper subspace, let alone a G-invariant one.

Example 1.23. The irreducible complex representations of $\mathbb{Z}_n$ (that is, the representations $\psi : \mathbb{Z}_n \to \mathbb{C}$) are the maps $1 \mapsto \xi$, where ξ is an nth root of unity (a complex root of the polynomial $x^n - 1$).

Example 1.24. For $t \in \mathbb{Z}$, let $\varphi(t) := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. The map $\varphi : \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ is a non-irreducible representation of $\mathbb{Z}$.

Proof. The map φ is indeed a representation, as

$$\varphi(a)\varphi(b) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix} = \varphi(ab).$$

Consider $\{(\alpha, 0) \mid \alpha \in \mathbb{Z}\}$, a subspace of $\mathbb{C}^2$. Under any element of the representation, $(\alpha, 0)$ is equal to itself:

$$(\alpha, 0) \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = (\alpha, 0).$$

Thus, $\{(\alpha, 0) \mid \alpha \in \mathbb{Z}\}$ is an invariant subspace, meaning that φ is not irreducible. $\square$

Showing that a given representation is reducible or irreducible is generally a difficult task. To show that a representation is reducible, we need to find a non-trivial proper G-invariant subspace. This can be easier said than done, especially in the case that we are working with an infinitely large algebraic structure. To show that a representation is irreducible, we often assume the contrary and reason to a contradiction, rather than perform case-exhaustive computation. These approaches will be used later in this thesis and are important to producing some key results.

Before ending our introductory chapter, we present two very powerful theorems

– one of Maschke and one of Artin-Wedderburn. Maschke's Theorem is a powerful tool in representation theory, giving us a way to break down certain representations into direct sums of subrepresentations. The proof is beyond the scope of this Independent Study, so we leave Maschke's Theorem without it (a proof may be found in [5]). Maschke's Theorem has multiple formulations, three of which are provided in Theorem 1.5.

Theorem 1.5 (Maschke)**.**

1. If V is a complex representation of a finite group G with W a subrepresentation, then there exists a subrepresentation U of V such that $V = W \oplus U$.

2. Let $\mathbb{F}$ be a field of characteristic k. Every representation of a finite group G over $\mathbb{F}$ such that $k \nmid |G|$ is a direct sum of irreducible representations.

3. If G is represented on $\mathbb{C}^n$ and W is G-invariant with respect to this representation, then there exists a subspace Z of $\mathbb{C}^n$ such that Z is G-invariant and $\mathbb{C}^n = W \oplus Z$ [4].

The Artin-Wedderburn Theorem is also given without proof and will be useful in showing the main results of the second and third chapters.

Definition 1.18 Let $\mathcal{A}$ be an algebra and $\mathcal{K}$ be a structure that acts on $\mathcal{A}$. If $\mathcal{A}$ decomposes as the direct sum of irreducible $\mathcal{K}$-invariant subspaces, then we say that $\mathcal{A}$ is **semisimple**.

Theorem 1.6 (Artin-Wedderburn)**.**

For an algebra $\mathcal{A}$ of finite dimension over a field $\mathbb{F}$, the following are equivalent:

1. *$\mathcal{A}$ is semisimple;*
2. $\sum_i \dim(V_i)^2 = \dim(\mathcal{A})$, *where the V_i's are irreducible representations of $\mathcal{A}$;*
3. *$\mathcal{A}$ is isomorphic to the direct sum of matrix algebras with entries in $\mathbb{F}$;*
4. *Any finite dimensional representation of $\mathcal{A}$ is completely reducible;*
5. *The regular representation of $\mathcal{A}$ is completely reducible.*

We are generally concerned with semisimplicity, and one formulation of Maschke's Theorem tells us about the semisimplicity of group algebras. Theorem 1.7 will be useful in showing the existence of a certain collection of semisimple subalgebras.

Theorem 1.7 (Maschke)**.**

Let G be a finite group and K a field of characteristic k. If $k \nmid |G|$, then KG, the group algebra of G over K, is semisimple.

CHAPTER 2

The Planar Rook Algebra

In this chapter, we construct the planar rook algebra from the planar rook monoid. Using tools from representation theory and the study of actions, we decompose the planar rook algebra into the direct sum of P_n-invariant irreducible subspaces, which helps us attain the main result of the chapter, that the planar rook algebra is isomorphic to the direct sum of matrix algebras.

A good amount of the material in this section comes from [1], although we will still make a point to cite [1] when necessary.

2.1 The Planar Rook Monoid

We looked briefly at the rook monoid. We now explore the *planar* rook monoid. Recall that the **rook monoid** R_n is the set of $n \times n$ matrices with entries in $\mathbb{F}_2$ such that there is at most one 1 in each column and in each row together with matrix multiplication.

For example,

$$R_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

To each element of R_n we associate a **rook diagram** d, which is a bipartite graph on two rows of vertices, where the specified independent sets are the top vertices and the bottom vertices. We require that each vertex has degree at most 1. An example of a rook diagram in R_4 can be seen in Figure 2.1.

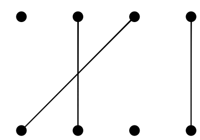


Figure 2.1: A diagram in R_4

To associate a matrix to a rook diagram, we first label the top vertices of the diagram from left to right with the numbers 1 to n, and then do the same to the bottom vertices. If vertex i on the top is adjacent to vertex j on the bottom, then the (i, j)-entry of the corresponding matrix will be a 1. Otherwise, it will be a 0. For example, the matrix that corresponds to the rook diagram in Figure 2.1 can be seen in Figure 2.2.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 2.2: The matrix associated to the diagram in Figure 2.1

The diagram in Figure 2.1 is part of the rook monoid R_n, but it is not part of the **planar rook monoid** P_n, a submonoid given by the set of all planar rook diagrams. A planar rook diagram is akin to a planar graph: it must be drawn without crossed edges and without drawing an edge outside of the rectangle bounded by the vertices labeled 1 and n. In terms of matrices,

$$P_n = \{M \in R_n \mid M \text{ is semi-echeloned}\}.$$

A **semi-echeloned** matrix M is one that is almost in reduced row echelon form, but not quite: we only require that the 1's in M are all to the left of any 1's below them. Figure 2.3 shows three planar rook diagrams and their associated matrices.

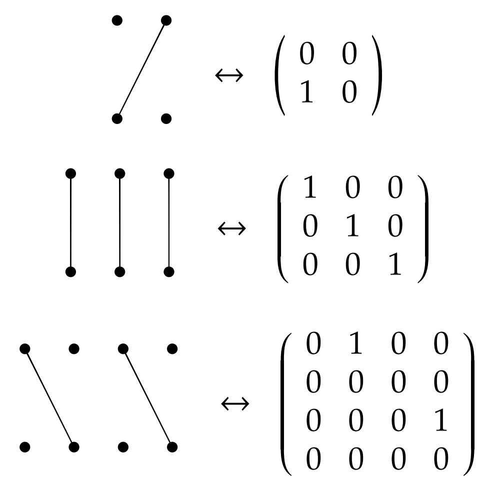
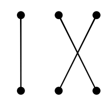
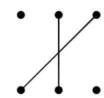


Figure 2.3: Three planar rook diagrams and their associated matrices

In P_n, as well as in R_n, matrix multiplication corresponds to diagram stacking. That is, if $d_1, d_2 \in R_n$ and M_1 and M_2 are the associated matrices for d_1 and d_2, respectively, then M_1M_2 is the matrix associated with d_1d_2, where d_1d_2 is obtained by stacking d_1 on top of d_2, deleting "dead ends," and connecting complete paths.

Example 2.1. In R_3, let

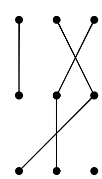
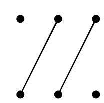
$d_1 =$ 

$d_2 =$ 

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

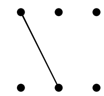
$$M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

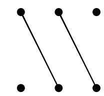
Then,

$d_1 d_2 =$  $=$ 

$$M_1 M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 2.2. If we restrict ourselves to P_3, we can look at

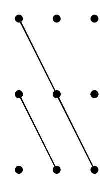
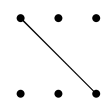
$d_1 =$ 

$d_2 =$ 

$$M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then,

$d_1 d_2 =$  $=$ 

$$M_1 M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example 2.2 illustrates that if d_1 and d_2 are planar rook diagrams, then $d_1 d_2$ is a planar rook diagram, just as the product of semi-echeloned matrices is again a semi-echeloned matrix. Matrix multiplication is associative, and certainly the $n \times n$ identity matrix I_n is a semi-echeloned matrix. Together, these facts show that P_n is indeed a submonoid of R_n.

Theorem 2.1.

The cardinality of P_n is

$$|P_n| = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Proof. The first equality is the easier to see. Given an empty diagram in P_n, we have n top vertices and n bottom vertices. We may choose either $1, 2, \ldots,$ or n vertices on the top to have degree 1 and the same number on the bottom to have degree 1. That is, we have $\binom{n}{k}$ vertices on the top with degree 1 and $\binom{n}{k}$ vertices on the bottom with degree 1. Because d is planar, there is only one way to draw the edges. This is because the matrix associated with d must be semi-echeloned. Thus,

$$|P_n| = \sum_{k=0}^{n} \binom{n}{k}^2.$$

The second equality takes a bit more thinking. Suppose we are choosing equal-sized teams from a pool of $2n$ people. We may choose n people to be on team red, and the remaining n people are then forced to be on team blue. This is represented as $\binom{2n}{n}$. An equivalent way to choose the teams is to first split the participants into equal sets of n people. Then, from the first n, choose k to be on team red and put the other $n-k$ on team blue. From the next n, choose k to be on team blue and put the other $n-k$ on team red. Then, each team has n members. We can do this with k equal to $1, 2, \ldots,$ or n. This is represented by

$$\sum_{k=0}^{n} \binom{n}{k}^2.$$

Thus,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Hence,

$$|P_n| = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

□

We end this section with a recording of the full sets of diagrams in P_2 and P_3 (Figures 2.4 and 2.5, respectively). We will not give P_4 and above, as $|P_n|$ grows rather quickly – already, $|P_4| = \binom{2\times 4}{4} = 70$. The sequence $a_n = \binom{2n}{n}$ is called the sequence of central binomial coefficients and is sequence A000984 in The On-Line Encyclopedia of Integer Sequences [6].

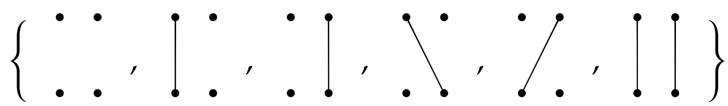
$P_2 =$ 

Figure 2.4: The elements of P_2

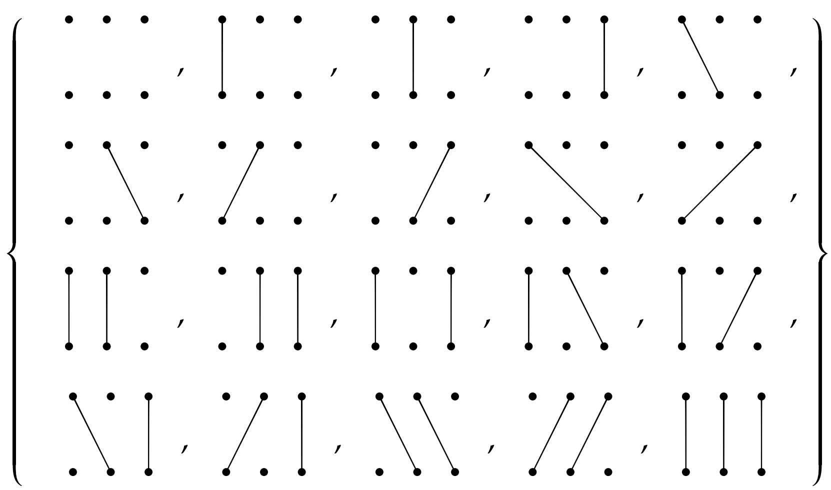
$P_3 =$ 

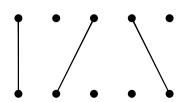
Figure 2.5: The elements of P_3

2.2 The Action of P_n

We are working toward a definition of the planar rook algebra. The next step is to look at how P_n acts on a set. To do so, we need to introduce some notation.

Let $d \in P_n$. The set $\tau(d)$ is the set is vertices in the top row of d that have degree 1. Similarly, $\beta(d)$ is the set of vertices in the bottom row of d that have degree 1. As long as d is a planar rook diagram, $\tau(d)$ and $\beta(d)$ uniquely determine d (of course, this is not the case if d is not planar).

Example 2.3. Consider $d \in P_5$ such that $\beta(d) = \{1, 2, 5\}$ and $\tau(d) = \{1, 3, 4\}$. The only diagram in P_n that satisfies these top and bottom sets is

$d =$  .

Essentially, a planar rook diagram d is an injective function from $\beta(d)$ to $\tau(d)$. If we take the diagram in Example 2.3, we get

$$d(1) = 1$$

$$d(2) = 3$$

$$d(5) = 4$$

as the corresponding injective function. Keep in mind that the injective functions corresponding to the diagrams map from bottom to top and not from top to bottom.

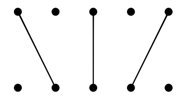
The relationship between $\beta(d)$ and $\tau(d)$ is helpful in defining an action of P_n. The following definition is not the final action for which we are going, but it is indeed an action.

Definition 2.1 Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq \{1, 2, \ldots, n\}$. If $d \in P_n$ and $S \subseteq \beta(d)$, we define

$$d(S) = \{d(s_1), d(s_2), \ldots, d(s_k)\}.$$

If S is not a subset of $\beta(d)$, then $d(S)$ is left undefined.

Example 2.4. Let

$d =$  and $S = \{2, 4\}$.

Then, $d(S) = \{1, 5\}$.

Using diagrams as injective functions, we can look at the action of P_n on an abstract vector space. Let V^n be the 2^n-dimensional complex vector space with basis

$$\{\vec{v}_S \mid S \subseteq \{1, 2, \ldots, n\}\}.$$

The basis vectors of V^n are indexed by subsets of $\{1, 2, \ldots, n\}$, which is why V^n has dimension 2^n, as there are 2^n such subsets. For example, $\vec{v}_{\{1,3,4\}}$ and $\vec{v}_{\{1,2\}}$ are basis vectors of V^4.

From linear algebra, we are used to the fact that the span of the basis of a vector space is equal to the entire vector space. However, we are usually only considering the real span, or the $\mathbb{R}$-span. Here, we will consider the complex span, or the $\mathbb{C}$-span. That is,

$$V^n = \mathbb{C}\text{-span}\{\vec{v}_S \mid S \subseteq \{1, 2, \ldots, n\}\}.$$

Simply put, our scalars are coming from $\mathbb{C}$, rather than from $\mathbb{R}$.

We can see that P_n acts on V^n in a natural way. An element $d \in P_n$ acts on the basis elements of V^n by

$$(\vec{v}_S)^d = \begin{cases} \vec{v}_{d(S)}, & \text{if } S \subseteq \beta(d) \\ \vec{0}, & \text{otherwise.} \end{cases} \tag{2.1}$$

This action extends linearly to general elements of V^n. This means that it behaves like a linear map. Let $\vec{w} \in V^n$. Then,

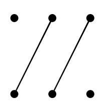
$$\vec{w} = \sum_{S \subseteq \{1,2,\ldots n\}} \lambda_S \vec{v}_S$$

is some linear combination of the basis elements of V^n. Because the action extends linearly, if we have $d \in P_n$ acting on $\vec{w}$, then

$$(\vec{w})^d = \left(\sum_{S \subseteq \{1,2,\ldots n\}} \lambda_S \vec{v}_S \right)^d = \sum_{S \subseteq \{1,2,\ldots n\}} (\lambda_S \vec{v}_S)^d = \sum_{S \subseteq \{1,2,\ldots n\}} \lambda_S (\vec{v}_S)^d = \sum_{S \subseteq \{1,2,\ldots n\}} \lambda_S \vec{v}_{d(S)}.$$

Example 2.5. Let us look at a concrete example of this action. Consider V^3, which has basis

$$\{\vec{v}_\emptyset, \vec{v}_{\{1\}}, \vec{v}_{\{2\}}, \vec{v}_{\{3\}}, \vec{v}_{\{1,2\}}, \vec{v}_{\{1,3\}}, \vec{v}_{\{2,3\}}, \vec{v}_{\{1,2,3\}}\}.$$

Let $d =$  . Here, $\tau(d) = \{2,3\}$ and $\beta(d) = \{1,2\}$. Then,

$$\begin{aligned}
(\vec{v}_\emptyset)^d &= \vec{v}_\emptyset \\
(\vec{v}_{\{1\}})^d &= \vec{v}_{\{2\}} \\
(\vec{v}_{\{2\}})^d &= \vec{v}_{\{3\}} \\
(\vec{v}_{\{3\}})^d &= \vec{0} \\
(\vec{v}_{\{1,2\}})^d &= \vec{v}_{\{2,3\}} \\
(\vec{v}_{\{1,3\}})^d &= \vec{0} \\
(\vec{v}_{\{2,3\}})^d &= \vec{0} \\
(\vec{v}_{\{1,2,3\}})^d &= \vec{0}.
\end{aligned}$$

Given a basis vector $\vec{v}_S$, a diagram d such that $S \subseteq \beta(d)$ will only map $\vec{v}_S$ to a basis vector $\vec{v}_T$ with $|T| = |S|$. It follows that if we restrict S to a certain size, say k, and look at

$$V_k^n = \mathbb{C}\text{-span}\left\{\vec{v}_S \mid S \subseteq \{1, 2, \ldots, n\} \text{ and } |S| = k\right\},$$

then we get a P_n-invariant subspace of V^n. It is a subspace because the sum of $\vec{w}$ and $\vec{z}$ in V_k^n can be written in terms of the basis vectors $\vec{v}_S$, as vector addition does not alter $|S|$. That is, if

$$\vec{w} = \sum \omega_S \vec{v}_S \quad \text{and} \quad \vec{z} = \sum \zeta_S \vec{v}_S,$$

where $\omega_S, \zeta_S \in \mathbb{C}$, then,

$$\begin{aligned} \vec{w} + \vec{z} &= \sum \omega_S \vec{v}_S + \sum \zeta_S \vec{v}_S \\ &= \sum (\omega_S + \zeta_S) \vec{v}_S, \end{aligned}$$

where all sums run over $S \subseteq \{1, 2, \ldots, n\}$ with $|S| = k$. Similarly, scalar multiplication does not alter $|S|$ in the basis vectors $\vec{v}_S$. The subspace is P_n-invariant because the action of $d \in P_n$ does not change $|S|$ for $S \subseteq \beta(d)$. That is, $|d(S)| = |S|$ for $S \subseteq \beta(d)$. This is a result of the fact that d can be thought of as an injective function from $\beta(d)$ to $\tau(d)$. Thus, V_k^n is P_n-invariant.

Theorem 2.2 ([1]).
For all $n \geq 0$ and $0 \leq k \leq n$, we have

1. *As P_n-modules, $V_k^n \cong V_m^n$ if and only if $k = m$,*

2. *V_k^n is irreducible,*

3. *V^n decomposes as*

$$V^n \cong \bigoplus_{k=0}^{n} V_k^n,$$

with each V_k^n appearing exactly once.

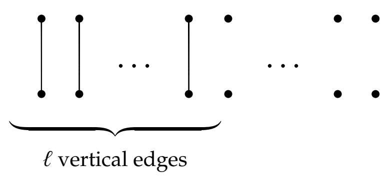
Proof. 1. We know that, as vector spaces, $V_k^n \cong V_m^n$ if and only if $k = m$, as V_k^n and

V^n_m are finite-dimensional vector spaces. Thus, V^n_k and V^n_m are not isomorphic as P_n-modules if $k \neq m$.

The initial reaction is that the claim holds because of the variation in dimension of the different V^n_k's. However, recall that $\binom{n}{k} = \binom{n}{n-k}$, so V^n_k and V^n_{n-k} are of equal dimension. Thus, we need a different approach, as even if V^n_k and V^n_m are isomorphic as vector spaces, they may not be isomorphic as P_n-modules.

For V^n_k and V^n_m to be isomorphic, the way that P_n acts on each subspace must be the same. It does not make sense for, say, the Klein four-group K_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ to behave differently under the action of a group G, as, regardless of what they are called, they are the "same" group.

Consider the following element of P_n:

$p_\ell =$ 

,

where p_ℓ has ℓ vertical edges between the first ℓ pairs of vertices and is empty everywhere else. The set of p_ℓ for $0 \leq \ell \leq n$ acts differently on V^n_k and V^n_{n-k}. If $\ell \geq k$, then p_ℓ acts like the identity diagram on all $\vec{v} \in V^n_k$. However, if $\ell < k$, then p_ℓ acts on $\vec{v} \in V^n_k$ by sending it to $\vec{0}$, because $S \not\subseteq \beta(p_\ell)$ when $\ell < |S| = k$. Essentially, we are trying to fit a finite set into a smaller finite set, which cannot be done. Thus, as P_n does not act on V^n_k in the same way that it acts on V^n_{n-k}, we have that V^n_k and V^n_{n-k} are not isomorphic as P_n-modules.

2. We want to show that V^n_k is irreducible. That is, we want to show that it has no non-trivial proper P_n-invariant subspaces. To do so, we show that any non-trivial P_n-invariant subspace W of V^n_k contains all basis vectors of V^n_k, meaning that $W = V^n_k$.

Suppose that V_k^n is not irreducible. Then, there exists a non-trivial proper subspace W of V_k^n that is P_n-invariant. Let $\vec{0} \neq \vec{w} \in W$, and consider $\vec{w}$ as a linear combination of basis vectors:

$$\vec{w} = \sum_{|S|=k} \lambda_S \vec{v}_S,$$

where $\lambda_S \in \mathbb{C}$. As $\vec{w}$ is non-zero, we have that there exists S such that λ_S is non-zero. Denote this S by S^*. Let $d \in P_n$ be the unique planar rook diagram with $\beta(d) = \tau(d) = S^*$. Then, we have that $(\vec{w})^d = \lambda_{S^*}\vec{v}_{S^*}$. This equality is a little tricky to see. Remember that for all $\vec{v}_S \in W$, the size of S is k. We also know that $(\vec{v}_S)^d = \vec{0}$ in all cases that $S \not\subseteq \beta(d)$. Because we are working with d such that $\beta(d) = \tau(d) = S^*$, there is only one $\vec{v}_S$ that does not map to $\vec{0}$ under d, namely $\vec{v}_{S^*}$. Moreover, $\vec{v}_{S^*}$ is mapped to itself.

The fact that $(\vec{w})^d = \lambda_{S^*}\vec{v}_{S^*}$ implies that $\vec{v}_{S^*} \in W$, as W is P_n-invariant and we can easily undo the multiplication by non-zero λ_{S^*}.

Now we show that we can act on $\vec{v}_{S^*}$ to obtain the rest of the basis vectors. We do this similarly to how we obtained $\vec{v}_{S^*} \in W$. Let $T \subseteq \{1, 2, \ldots, n\}$ such that $|T| = k$, and let b be the unique planar rook diagram with $\beta(b) = S^*$ and $\tau(b) = T$. Then,

$$\left(\vec{v}_{S^*}\right)^b = \vec{v}_{b(S^*)} = \vec{v}_T.$$

Thus, $\vec{v}_T \in W$. Hence, all basis vectors of V_k^n are elements of W, implying $W = V_k^n$. Therefore, V_k^n is irreducible.

3. Note that each $\vec{v}_S$ appears exactly once in the V_k^n's, because of the variation in the value of k. The isomorphism is the one that maps $\vec{v}_S \in V^n$ to $\vec{v}_S \in \bigoplus_{k=0}^n V_k^n$.

That is, letting φ be the map,

$$\varphi(\vec{v}_S) = \underbrace{\vec{0} + \vec{0} + \ldots + \vec{0} + \vec{v}_S + \vec{0} + \ldots + \vec{0} + \vec{0}}_{n+1 \text{ terms}}.$$

The bases are put in bijection by this map, as $\vec{0}+\ldots+\vec{0}+\vec{v}_S+\vec{0}+\ldots+\vec{0} \in \bigoplus_{k=0}^{n} V_k^n$ is mapped to by $\vec{v}_S \in V^n$, and if $\vec{0}+\ldots+\vec{0}+\vec{v}_S+\vec{0}+\ldots+\vec{0} = \vec{0}+\ldots+\vec{0}+\vec{v}_T+\vec{0}+\ldots+\vec{0}$, then $\vec{v}_S = \vec{v}_T$. The map φ is also a homomorphism, as

$$\begin{aligned}
\varphi(\vec{v}_S) + \varphi(\vec{v}_T) &= (\vec{0} + \ldots + \vec{0} + \vec{v}_S + \vec{0} + \ldots + \vec{0}) + (\vec{0} + \ldots + \vec{0} + \vec{v}_T + \vec{0} + \ldots + \vec{0}) \\
&= \vec{0} + \ldots + \vec{0} + \vec{v}_S + \vec{v}_T + \vec{0} + \ldots + \vec{0} \\
&= \varphi(\vec{v}_S + \vec{v}_T).
\end{aligned}$$

The structure of the module is preserved, as well, considering

$$\begin{aligned}
\lambda\varphi(\vec{v}_S) &= \lambda(\vec{0} + \ldots + \vec{0} + \vec{v}_S + \vec{0} + \ldots + \vec{0}) \\
&= \vec{0} + \ldots + \vec{0} + \lambda\vec{v}_S + \vec{0} + \ldots + \vec{0} \\
&= \varphi(\lambda\vec{v}_S).
\end{aligned}$$

Thus, the result holds:

$$V^n \cong \bigoplus_{k=0}^{n} V_k^n.$$

$\square$

2.3 The Planar Rook Algebra

With fairly comprehensive knowledge of the planar rook monoid, we can look at the algebra that arises from P_n.

Consider

$$\mathbb{C}P_n = \mathbb{C}\text{-span}\{d \mid d \in P_n\} = \left\{\sum_{d\in P_n} \lambda_d d \mid \lambda_d \in \mathbb{C}\right\}.$$

We call $\mathbb{C}P_n$ the **planar rook algebra**, and it is a vector space of dimension $|P_n| = \binom{2n}{n}$. The planar rook algebra consists of all linear combinations of planar rook diagrams. We know how to multiply diagrams, and can extend this linearly, but the ideas of addition and scalar multiplication make little sense geometrically and must be allowed to be abstract. The element $(2 + 3i)d$, for example, where $d \in P_n$, is not to be drawn, but understood simply by its definition: the scalar multiple of d by $(2 + 3i)$. Similarly, there is no geometric equivalent to $d_1 + d_2$. It is only to be considered in the abstract. That being said, nd, where $n \in \mathbb{N}$, is equivalent to

$$\underbrace{d + d + \ldots + d + d}_{n \text{ times}}.$$

The planar rook algebra is indeed an algebra, as it is a vector space and the algebra's binary operation is stacking. Recall that a vector space satisfies 8 properties, which $\mathbb{C}P_n$ does:

1. Commutativity of vector addition
2. Associativity of vector addition
3. Existence of an additive identity
4. Existence of additive inverses
5. Associativity of scalar multiplication
6. Distributivity of scalars over vectors
7. Distributivity of vectors over scalars

8. Existence of a scalar identity

Items 1 and 2 are naturally satisfied. As we stated earlier, there is no geometric equivalent to $d_1 + d_2$. It is only to be considered in the abstract. The zero vector is the additive identity. Item 4 is satisfied by noting that for $\vec{v} \in \mathbb{C}P_n$, we have that $-\vec{v}$ is its additive inverse. The scalars are complex numbers, and because $\lambda\vec{v}$ is in its most simplified form, $\alpha(\lambda\vec{v}) = (\alpha\lambda)\vec{v}$. We will not go through 6 and 7, but note that they also follow. The scalar identity is 1.

We now have that $\mathbb{C}P_n$ is a vector space. The criteria for $\mathbb{C}P_n$ to be an algebra are met by definition. That is, we have constructed $\mathbb{C}P_n$ in such a way that it satisfies the axioms of an algebra.

It is important to note the distinction between the diagram with no edges in $\mathbb{C}P_n$ (the **empty diagram**) and the zero vector $\vec{0}$ of $\mathbb{C}P_n$. The empty diagram is a basis element of $\mathbb{C}P_n$, while $\vec{0}$ is the linear combination of vectors where all scalars involved are zero.

As $\mathbb{C}P_n$ and P_n are very much related, the latter being a basis for the former, we can look at how they interact. Let $d \in P_n$ and $\vec{v} \in \mathbb{C}P_n$. There is a natural action of P_n on $\mathbb{C}P_n$ given by

$$(\vec{v})^d = d\vec{v} = d\left(\sum_{b\in P_n} \lambda_b b\right) = \sum_{b\in P_n} \lambda_b db.$$

The **rank** of a planar rook diagram d is the number of edges in d. Equivalently, it is the number of ones in the associated matrix of d. If $d_1, d_2 \in P_n$, we have that

$$\mathrm{rank}(d_1 d_2) \leq \min(\mathrm{rank}(d_1), \mathrm{rank}(d_2)).$$

This property, that rank does not increase through multiplication, should be fairly intuitive. Given $d_1, d_2 \in P_n$, the number of edges in d_1d_2 is the size of the set $\beta(d_1) \cap \tau(d_2)$, and we know from basic set theory that $|A \cap B| \leq \min(|A|, |B|)$.

We can use the non-increasing rank property to find P_n-invariant subspaces of $\mathbb{C}P_n$. Consider

$$X_k^n = \mathbb{C}\text{-span}\{d \in P_n \mid \operatorname{rank}(d) \leq k\}.$$

Each X_k^n is a P_n-invariant subspace of $\mathbb{C}P_n$. We can also see, however, that

$$X_0^n \subseteq X_1^n \subseteq \ldots \subseteq X_{n-1}^n \subseteq X_n^n.$$

While these are P_n-invariant subspaces, the fact that they form such a chain shows us that they are not irreducible.

The basis $\{d \mid d \in P_n\}$ of $\mathbb{C}P_n$ is the most obvious one and the one we used to define $\mathbb{C}P_n$ in the first place. However, to obtain our desired result, that the planar rook algebra is isomorphic to the direct sum of matrix algebras, we must change basis. To do so, we need a little bit of terminology and notation.

Definition 2.2 Let $d_1, d_2 \in P_n$. We say $d_1 \subseteq d_2$ if the edges of d_1 are a subset of the edges of d_2; equivalently, if

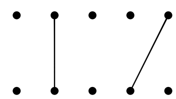
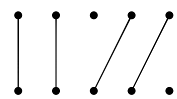
$$[M_1(i, j)] \subseteq [M_2(i, j)],$$

where

$$[M_k(i, j)] = \{(i, j) \mid \text{the } (i, j)\text{-entry of } M_k \text{ is } 1\},$$

and M_k is the rook matrix associated with d_k.

Example 2.6. For example,

$d_1 =$  $\subseteq$  $= d_2,$

with

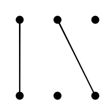
$$[M_1(i,j)] = \{(2,2),(5,4)\} \subseteq \{(1,1),(2,2),(4,3),(5,4)\} = [M_2(i,j)].$$

If $d_1 \subseteq d_2$, we denote $\mathrm{rank}(d_2) - \mathrm{rank}(d_1)$ by $|d_2 \backslash d_1|$. This, of course, is the difference in edge count between d_1 and d_2.

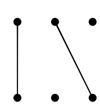
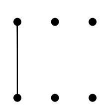
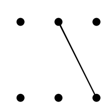
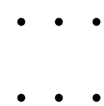
For $d \in P_n$, define

$$x_d = \sum_{b \subseteq d} (-1)^{|d \backslash b|} b. \tag{2.2}$$

Example 2.7. Let

$d =$  .

Then

$x_d =$  $-$  $-$  $+$  .

We claim that $\{x_d \mid d \in P_n\}$ forms a basis for $\mathbb{C}P_n$. Rank gives us a way to partially order planar rook diagrams. That is, $d_2 \leq d_1$ if $\mathrm{rank}(d_2) < \mathrm{rank}(d_1)$. More generally, a partial ordering on a set is an ordering that satisfies the reflexive, transitive, and antisymmetric properties. We are familiar with the first two from equivalence relations. An ordering $\leq$ is **antisymmetric** if, given $a \leq b$ and $b \leq a$, we have that $a = b$.

A total ordering on a set is one that is antisymmetric, transitive, and connex. **Connex** means that given any two elements of the set a and b, we have that $a \leq b$ or $b \leq a$. Intuitively, this is why the ordering is *total*, because any two elements are related in some way.

Given a partial ordering on a finite set, we can extend it to a total ordering, often in multiple ways. We simply need to declare that elements that were previously unrelated now have some sort of relation so as to satisfy connexity.

Example 2.8. Take the power set of $\{1,2,3\}$ ordered by subset inclusion. This is a partial ordering. To extend to a total ordering, declare that

$$\emptyset \preceq \{1\} \preceq \{2\} \preceq \{3\} \preceq \{1,2\} \preceq \{1,3\} \preceq \{2,3\} \preceq \{1,2,3\}.$$

This ordering is total and preserves the partial ordering given by subset inclusion.

We claim that if we extend the partial ordering on P_n given by rank to any total ordering, then a transition matrix M from $\{d \mid d \in P_n\}$ to $\{x_d \mid d \in P_n\}$ is upper triangular and has eigenvalue 1 with multiplicity $\binom{2n}{n}$. Note that we specifically said *a* transition matrix, as the transition matrix based on one total ordering may not be the same as the transition matrix based on another total ordering.

Before we prove our claim, let us recall from linear algebra the notion of a transition matrix. The standard basis for $\mathbb{R}^3$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

denoted $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$. Note that

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

is also a basis for $\mathbb{R}^3$, as

$$\vec{e_1}, \vec{e_2}, \vec{e_3} \in \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

If we denote our new basis $\{\vec{k_1}, \vec{k_2}, \vec{k_3}\}$, then

$$\begin{aligned} \vec{k_1} &= 2\vec{e_1} + \vec{e_3} \\ \vec{k_2} &= 3\vec{e_2} \\ \vec{k_3} &= -\vec{e_3}. \end{aligned}$$

Thus, the transition matrix from our basis of $\vec{e_i}$'s to our basis of $\vec{k_i}$'s is

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

as $M\vec{e_i} = \vec{k_i}$.

Theorem 2.3 ([1]).

The set $\{x_d \mid d \in P_n\}$ forms a basis for $\mathbb{C}P_n$. Furthermore, if we extend the partial ordering on P_n given by rank to any total ordering, then a transition matrix M from $\{d \mid d \in P_n\}$ to $\{x_d \mid d \in P_n\}$ is upper triangular and has eigenvalue 1 with multiplicity $\binom{2n}{n}$.

Proof. We begin by showing that $\{x_d \mid d \in P_n\}$ forms basis for $\mathbb{C}P_n$. Certainly, the x_d are linearly independent. Consider, noting that the next sum runs over proper subdiagrams,

$$x_d = d + \sum_{b \subset d} (-1)^{|d \setminus b|} b.$$

Thus,

$$d = x_d - \sum_{b \subset d} (-1)^{|d \setminus b|} b = x_d - \sum_{b \subset d} x_b.$$

Thus, $d \in \{x_d \mid d \in P_n\}$. This shows that $\{x_d \mid d \in P_n\}$ is a basis for $\mathbb{C}P_n$.

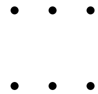
The fact that M is upper triangular follows from the partial ordering given by rank. For $d \in P_n$, the expansion x_d is a sum/difference of diagrams with ranks less than or equal to d. As

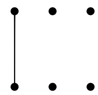
$$x_d = d - \sum_{b \subset d} (-1)^{|d \setminus b|} b,$$

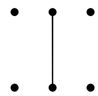
we have that each diagonal entry in M is a 1. As each diagram not equal to d in x_d has rank less than that of d, nonzero entries in the transition matrix must only occur above the 1's on the diagonal. Thus, M is upper-triangular. Hence, M has eigenvalue 1 with multiplicity $\binom{2n}{n}$, the size of P_n and the size of the matrix.

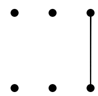
□

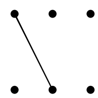
Example 2.9. Let us work through an example in P_3. We will look at a transition matrix from $\{d \mid d \in P_3\}$ to $\{x_d \mid d \in P_3\}$. First, we must extend the partial order given by rank and determine x_d for each $d \in P_3$. Remember that there are $\binom{2 \cdot 3}{3} = 20$ such d. Thus, we will end up with a 20×20 matrix. We will order as follows ($d_i \leq d_j$ if $i \leq j$). The x_d for each $d \in P_3$ are

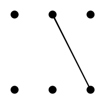
$d_1 =$  $\mapsto d_1 = x_{d_1}$

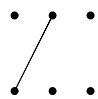
$d_2 =$  $\mapsto d_2 - d_1 = x_{d_2}$

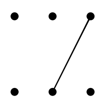
$d_3 =$  $\mapsto d_3 - d_1 = x_{d_3}$

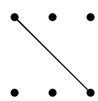
$d_4 =$  $\mapsto d_4 - d_1 = x_{d_4}$

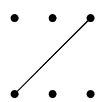
$d_5 =$  $\mapsto d_5 - d_1 = x_{d_5}$

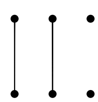
$d_6 =$  $\mapsto d_6 - d_1 = x_{d_6}$

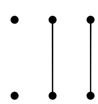
$d_7 =$  $\mapsto d_7 - d_1 = x_{d_7}$

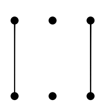
$d_8 =$  $\mapsto d_8 - d_1 = x_{d_8}$

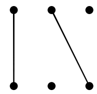
$d_9 =$  $\mapsto d_9 - d_1 = x_{d_9}$

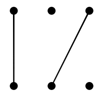
$d_{10} =$  $\mapsto d_{10} - d_1 = x_{d_{11}}$

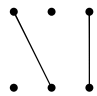
$d_{11} =$  $\mapsto d_{11} - d_3 - d_2 + d_1 = x_{d_{11}}$

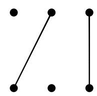
$d_{12} =$  $\mapsto d_{12} - d_4 - d_3 + d_1 = x_{d_{12}}$

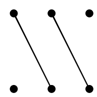
$d_{13} =$  $\mapsto d_{13} - d_4 - d_2 + d_1 = x_{d_{13}}$

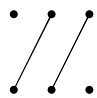
$d_{14} =$  $\mapsto d_{14} - d_6 - d_2 + d_1 = x_{d_{14}}$

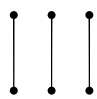
$d_{15} =$  $\mapsto d_{15} - d_7 - d_2 + d_1 = x_{d_{15}}$

$d_{16} =$  $\mapsto d_{16} - d_5 - d_4 + d_1 = x_{d_{16}}$

$d_{17} =$  $\mapsto d_{17} - d_7 - d_4 + d_1 = x_{d_{17}}$

$d_{18} =$  $\mapsto d_{18} - d_6 - d_5 + d_1 = x_{d_{18}}$

$d_{19} =$  $\mapsto d_{19} - d_8 - d_7 + d_1 = x_{d_{19}}$

$d_{20} =$  $\mapsto d_{20} - d_{13} - d_{12} - d_{11} + d_4 + d_3 + d_2 - d_1 = x_{d_{20}}$

Thus, the transition matrix from $\{d \mid d \in P_3\}$ to $\{x_d \mid d \in P_3\}$ is

$$\begin{pmatrix}
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

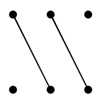
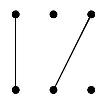
We now look at the action of P_n on $\mathbb{C}P_n$ with respect to our new basis $\{x_d \mid d \in P_n\}$. This action is similar to the one defined in Equation 2.1.

Let $b, d \in P_n$. Then,

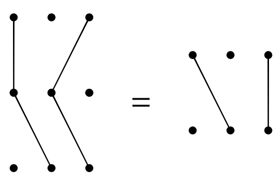
$$(x_b)^d = \begin{cases} x_{db}, & \text{if } \tau(b) \subseteq \beta(d), \\ \vec{0}, & \text{otherwise.} \end{cases} \tag{2.3}$$

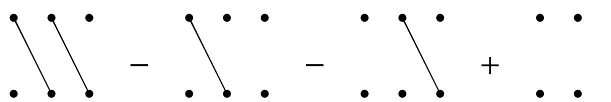
This action is clearly related to Equation 2.1, in which $(\vec{v}_S)^d = \vec{v}_{d(S)}$. When $\tau(b) \subseteq \beta(d)$, we have that $\tau(b') \subseteq \beta(d)$ for all $b' \subseteq b$. As d distributes over x_d, it maps each b' to db'. Thus, we are mapping x_b to x_{db}, as desired.

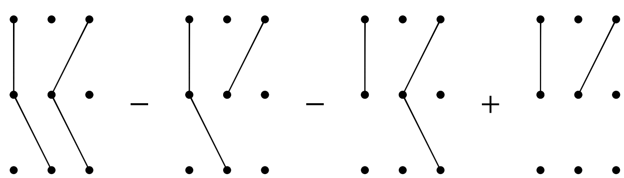
Example 2.10. Consider

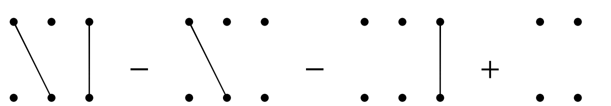
$b =$  and $d =$  in P_3.

Then,

$db =$ 

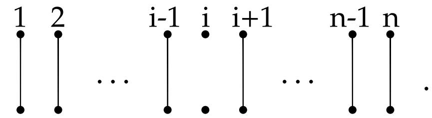
$x_b =$ 

$(x_b)^d =$ 

$=$ 

$= x_{db}$.

If $\tau(b) \nsubseteq \beta(d)$, we can look at why $(x_b)^d = \vec{0}$. Let $i \in \tau(b)$ such that $i \notin \beta(d)$. There must exist such an i by our supposition. Define $p_i \in P_n$ by

$p_i =$ 

That is, p_i is the identity diagram in P_n with the ith edge deleted. Because $i \notin \beta(d)$, we have that $p_i^d = d$. Furthermore, we have that

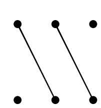
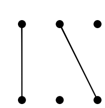
$$\begin{aligned} p_i x_b &= p_i \sum_{b' \subseteq b} (-1)^{|b \backslash b'|} b' \\ &= \sum_{b' \subseteq b} (-1)^{|b \backslash b'|} p_i b' \\ &= \sum_{\substack{b' \subseteq b \\ i \in \tau(b')}} (-1)^{|b \backslash b'|} p_i b' + \sum_{\substack{b' \subseteq b \\ i \notin \tau(b')}} (-1)^{|b \backslash b'|} p_i b'. \end{aligned}$$

If $i \notin \tau(b')$, then $p_i b' = b'$. When $i \in \tau(b')$, we have that $p_i b'$ is b' with the edge adjacent to the ith top vertex deleted. It is a simple combinatorial argument that there is a bijection between $\{b' \mid i \in \tau(b')\}$ and $\{b' \mid i \notin \tau(b')\}$ given by adding/removing the ith edge, that is, by multiplying by p_i. Because we are adding/removing an edge, $|b \backslash b'|$ changes by a factor of -1. Thus,

$$\sum_{\substack{b' \subseteq b \\ i \in \tau(b')}} (-1)^{|b \backslash b'|} p_i b' + \sum_{\substack{b' \subseteq b \\ i \notin \tau(b')}} (-1)^{|b \backslash b'|} p_i b' = \vec{0}.$$

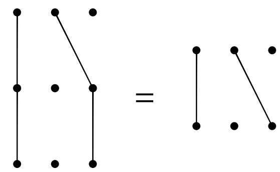
Hence, $(x_b)^d = (p_i x_b)^d = \vec{0}$.

Example 2.11. Consider

$b =$  and $d =$  in P_3.

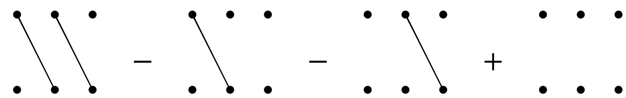
Then, $2 \in \tau(b)$ and $2 \notin \beta(d)$. So,

$(p_2)^d =$

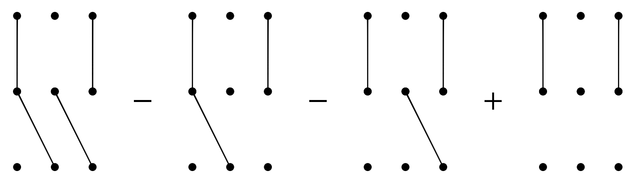


$= d$

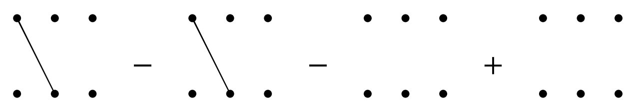
$x_b =$



$p_2 x_b =$



$=$



$= \vec{0}.$

Thus, $(x_b)^d = (x_b)^{dp_2} = (p_2 x_b)^d = \vec{0}$.

What really makes this result come out to $\vec{0}$ is the associativity of the action:

$$(p_2 x_b)^d = dp_2 x_b = \begin{cases} (dp_2)x_b = dx_b = x_{db} = (x_b)^d, \\ d(p_2 x_b) = d\vec{0} = \vec{0}. \end{cases}$$

For $b, d \in P_n$, we have that $\mathrm{rank}(b) = \mathrm{rank}(db)$ if and only if $\tau(b) \subseteq \beta(d)$. So, from the action of P_n on $\{x_b \mid b \in P_n\}$, we have that $(x_b)^d = \vec{0}$ if $\mathrm{rank}(b) \neq \mathrm{rank}(db)$. So,

$$W^{n,k} = \mathbb{C}\text{-span}\{x_b \mid \mathrm{rank}(b) = k\}$$

is a P_n-invariant subspace of $\mathbb{C}P_n$. Furthermore, the action of d on x_b does not alter

$\beta(b)$ if $\tau(b) \subseteq \beta(d)$. Then,

$$W_T^{n,k} = \mathbb{C}\text{-span}\{x_b \mid \text{rank}(b) = k, \beta(b) = T\} \tag{2.4}$$

is also a P_n-invariant subspace of $\mathbb{C}P_n$ for each such T. In fact, for any $T' \subseteq \{1, 2, \ldots, n\}$ such that $|T'| = |T| = k$, we have that

$$W_T^{n,k} \cong W_{T'}^{n,k}.$$

Finally, we see that

$$W_T^{n,k} \cong V_k^n,$$

because the action of P_n on x_d in $\mathbb{C}P_n$ is equivalent to that of P_n on $\vec{v}_{\tau(d)}$ in V^n . Let $S, T \subseteq \{1, \ldots, n\}$, where $|S| = |T|$. Define $x_{S,T} = x_d$, where d is the planar diagram with $\tau(d) = S$ and $\beta(d) = T$. Recall that $\tau(d)$ and $\beta(d)$ completely determine d, meaning that d is unique. Then, the isomorphism between $W_T^{n,k}$ and V_k^n is given by $x_{S,T} \mapsto \vec{v}_S$. This map is certainly one-to-one, as if $\vec{v}_{S_1} = \vec{v}_{S_2}$, we have that $S_1 = S_2$, implying $x_{S_1,T} = x_{S_2,T}$. It is also onto, considering if $\vec{v}_S \in V_k^n$, then $x_{S,T}$ will map to it. Indeed, the map is also a homomorphism due to linearity. Thus, the claim of isomorphism holds.

Theorem 2.4 ([1]).
The decomposition of $\mathbb{C}P_n$ into P_n-invariant subspaces is given by

$$\mathbb{C}P_n = \bigoplus_{k=0}^{n} W^{n,k} = \bigoplus_{k=0}^{n} \bigoplus_{|T|=k} W_T^{n,k} \cong \bigoplus_{k=0}^{n} \binom{n}{k} V_k^n := \bigoplus_{k=0}^{n} \underbrace{\left(V_k^n + V_k^n + \ldots + V_k^n\right)}_{\binom{n}{k} \text{ times}}.$$

Proof. The first equality is fairly clear. Certainly, $W^{n,k} = W^{n,k'}$ if and only if $k = k'$. Summing the $W^{n,k}$'s accounts for all possible values of k, resulting in $\mathbb{C}P_n$.

The second equality will be shown if we can justify that

$$W^{n,k} = \bigoplus_{|T|=k} W_T^{n,k}.$$

Here, we can see that $W_T^{n,k} = W_{T'}^{n,k}$ if and only if $T = T'$. Summing over all T's with size k is equivalent to looking at $W^{n,k}$, because all T's being accounted for is the same as looking at all x_d with $\mathrm{rank}(d) = k$. So, the second equality is justified.

The isomorphism

$$\bigoplus_{k=0}^{n} \bigoplus_{|T|=k} W_T^{n,k} \cong \bigoplus_{k=0}^{n} \binom{n}{k} V_k^n$$

remains to be shown. We need only show that

$$\bigoplus_{|T|=k} W_T^{n,k} \cong \binom{n}{k} V_k^n$$

to get our result.

Consider

$$\sum \lambda_1 x_{d_1} + \sum \lambda_2 x_{d_2} + \ldots + \sum \lambda_{\binom{n}{k}} x_{d_{\binom{n}{k}}} \in \bigoplus_{|T|=k} W_T^{n,k},$$

where $\sum \lambda_i x_{d_i} \in W_{T_i}^{n,k}$. Here, $|T_i| = k$ and $T_i = T_j$ if and only if $i = j$. Our indices run from 1 to $\binom{n}{k}$ because $\binom{n}{k}$ is the number of different T's with $|T| = k$ (from n vertices in a diagram d, we choose k vertices to be $\beta(d)$).

Consider

$$\sum \omega_1 \vec{v}_{S_1} + \sum \omega_2 \vec{v}_{S_2} + \ldots + \sum \omega_{\binom{n}{k}} \vec{v}_{S_{\binom{n}{k}}} \in \binom{n}{k} V_k^n,$$

where $\sum \omega_i \vec{v}_{S_i} \in V_k^n$. Our indices run from 1 to $\binom{n}{k}$, as there are $\binom{n}{k}$ copies of V_k^n.

In both summations, we are leaving out some notation. The $\sum$'s run over their own indices, and this is left out with the understanding that (1) they are implied and (2) including them would simply make things more crowded, as we would have to introduce a number of subscripts.

The isomorphism, then, is defined by

$$\sum \lambda_1 x_{d_1} + \sum \lambda_2 x_{d_2} + \ldots + \sum \lambda_{\binom{n}{k}} x_{d_{\binom{n}{k}}} \mapsto \sum \lambda_1 \vec{v}_{S_1} + \sum \lambda_2 \vec{v}_{S_2} + \ldots + \sum \lambda_{\binom{n}{k}} \vec{v}_{S_{\binom{n}{k}}}.$$

Showing that the map is a bijection is simple and omitted. That our map is a homomorphism holds due to linearity. Thus, the isomorphism holds. □

Theorem 2.5 ([1]).
The set $\left\{V_k^n \mid 0 \leq k \leq n\right\}$ is a complete set of irreducible $\mathbb{C}P_n$-modules.

Proof. Theorem 2.2 tells us that $\left\{V_k^n \mid 0 \leq k \leq n\right\}$ forms a complete set of irreducible subspaces. Theorem 2.4 tells us that the V_k^n's are $\mathbb{C}P_n$-modules. □

We now know what $\mathbb{C}P_n$ looks like in terms of the V_k^n's. By Theorems 2.2 and 2.4, we have that the planar rook algebra is semisimple (Definition 1.18). While the V_k^n's are familiar from reading this particular thesis, we would like to relate the structure of $\mathbb{C}P_n$ to something even more familiar.

Theorem 2.6 ([1]).
Let $S, T, U, V \subseteq \{1, 2, \ldots, n\}$ such that $|S| = |T|$ and $|U| = |V|$. Then,

$$x_{S,T}x_{U,V} = \begin{cases} x_{S,V}, & \text{if } T = U, \\ \vec{0}, & \text{otherwise.} \end{cases}$$

Proof. Recall that for $b, d \in P_n$, we have that

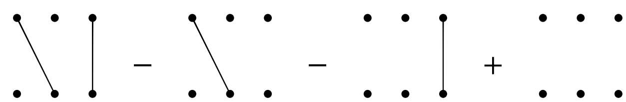
$$(x_b)^d = \begin{cases} x_{db}, & \text{if } \tau(b) \subseteq \beta(d), \\ \vec{0}, & \text{otherwise.} \end{cases}$$

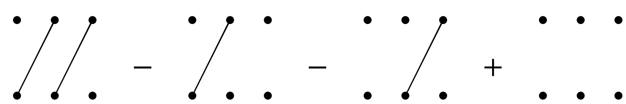
This was Equation 2.3 and essentially implies the theorem. If $T \neq U$, then either there exists $i \in T$ such that $i \notin U$ or $j \in U$ such that $j \notin T$. The same argument that showed Equation 2.3 holds shows that

$$x_{S,T}x_{U,V} = \begin{cases} x_{S,V}, & \text{if } T = U, \\ \vec{0}, & \text{otherwise,} \end{cases}$$

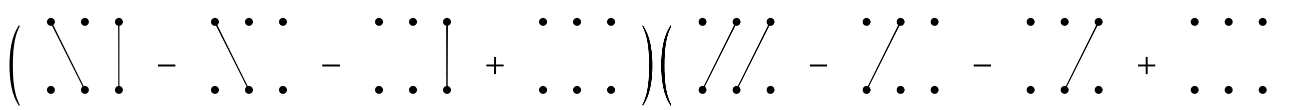
holds, as well. □

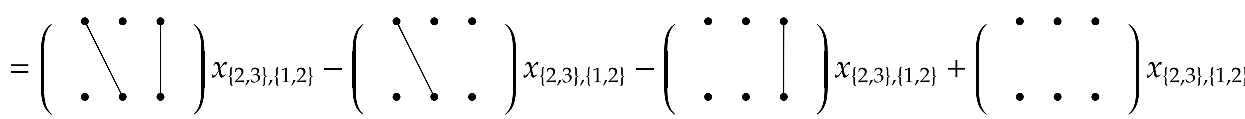
Example 2.12. Consider

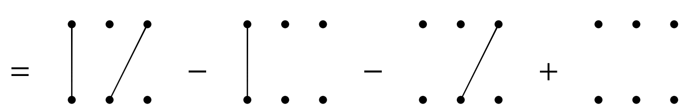
$x_{\{1,3\},\{2,3\}} =$  ,

$x_{\{2,3\},\{1,2\}} =$  $\in P_3$.

Then, $x_{\{1,3\},\{2,3\}}x_{\{2,3\},\{1,2\}} =$





 $- \vec{0} - \vec{0} + \vec{0}$

$= x_{\{1,3\},\{1,2\}}$.

The last three products are zero because $\tau(b) \not\subseteq \beta(d)$, as described in the action of d on x_b. That is, $\{2,3\}$ is not a subset of the bottom sets of any of the last three diagrams with which it is multiplied, meaning that the products are all zero.

When expressed in this manner, the behavior of x_d multiplication looks familiar. Theorem 2.6 tells us that $x_{S,T}$ behaves like $E_{i,j}$, the square matrix with a 1 in the (i,j)-entry and zeros everywhere else. That is, $E_{i,j} \times E_{k,\ell} = E_{i,\ell}$ when $j = k$ and zero otherwise. For example, in $M_{3\times 3}$,

$$E_{1,2} \times E_{2,3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = E_{1,3},$$

whereas

$$E_{1,2} \times E_{1,3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 2.7 ([1]).
The planar rook algebra $\mathbb{C}P_n$ is isomorphic to $\bigoplus_{k=0}^{n} \mathbf{Mat}\left(\binom{n}{k}, \binom{n}{k}\right)$, the algebra of $\binom{n}{k} \times \binom{n}{k}$ complex matrices. That is,

$$\mathbb{C}P_n \cong \bigoplus_{k=0}^{n} \mathbf{Mat}\left(\binom{n}{k}, \binom{n}{k}\right).$$

Proof. Certainly, $|\mathbb{C}P_n| = \left|\bigoplus_{k=0}^{n} \mathbf{Mat}\left(\binom{n}{k}, \binom{n}{k}\right)\right|$, as they are of equal dimension. The basis of $\mathbb{C}P_n$ is $\{x_d \mid d \in P_n\}$, which has size

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

The basis for $\mathbf{Mat}(\binom{n}{k}, \binom{n}{k})$ at which we are looking is $\left\{E_{i,j} \mid i, j \in \left\{1, \ldots, \binom{n}{k}\right\}\right\}$. This has size $\binom{n}{k}^2$, as this is the number of possible entries in which to place a single 1 in $I_{\binom{n}{k}}$. Thus, a basis for $\bigoplus_{k=0}^{n} \mathbf{Mat}\left(\binom{n}{k}, \binom{n}{k}\right)$ has

$$\sum_{k=0}^{n} \binom{n}{k}^2$$

elements.

The isomorphism comes from the fact that $\mathbb{C}P_n$ and $\bigoplus_{k=0}^{n} \mathbf{Mat}\left(\binom{n}{k}, \binom{n}{k}\right)$ are of the same dimension and have the same **relations**, a concept found in the study of (group) presentations. Essentially, we are working with words written in some generating set. The way that we determine equivalence between two words is via a relation. A relation in a group may be that $g^5 = e$ or that $ab = ba$. The relation in $\bigoplus_{k=0}^{n} \mathbf{Mat}\left(\binom{n}{k}, \binom{n}{k}\right)$ is that $E_{i,j} \times E_{k,\ell} = E_{i,\ell}$ if and only if $j = k$, else $\vec{0}$. The relation in $\mathbb{C}P_n$ is the same in the sense that if we replace E with x and i, j, k, ℓ with S, T, U, V, respectively, nothing has really changed (allowing that $\times$ refers to multiplication as defined in whichever space we are working). This fact, that two algebraic structures with the same dimension and relations are isomorphic, proves the theorem. □

In this chapter, we decomposed the planar rook algebra into the direct sum of P_n-invariant irreducible subspaces, showing that $\mathbb{C}P_n$ is semisimple. Alternatively, by using Theorem 2.7 in conjunction with Theorem 1.6 (3), we get that the planar rook algebra is semisimple.

CHAPTER 3

G-Edge-Coloring $\mathbb{C}P_n$

In this chapter, we look at what happens when we color the edges of the diagrams in $\mathbb{C}P_n$. Coloring an edge simply means assigning a label x to an edge, so that the edge in question may be thought of as having color x. Specifically, we will be coloring edges with elements of some group G. This allows us to redefine diagram multiplication, work toward a decomposition of the colored $\mathbb{C}P_n$ in the case that G is finite abelian, and observe the nature of the planar rook algebra when colored by finite nonabelian groups. The majority of the content in this chapter comes from [2], especially that which is in Section 3.3.

3.1 Coloring $\mathbb{C}P_n$

Let G be a group, and let

$$P_n(G) = \{d \in P_n \mid d \text{ is colored by } G\}.$$

Previously, our rook diagrams only had edges of one color. Now, we allow more colors, one for each element of G, and we distinguish between diagrams that are colored differently. Letting $G = \mathbb{Z}_2$ be our "palette", we have that $P_2(\mathbb{Z}_2)$ consists of the diagrams in Figure 3.1, where the color $z \in \mathbb{Z}_2$ is noted on the top vertex incident to the edge that it colors.

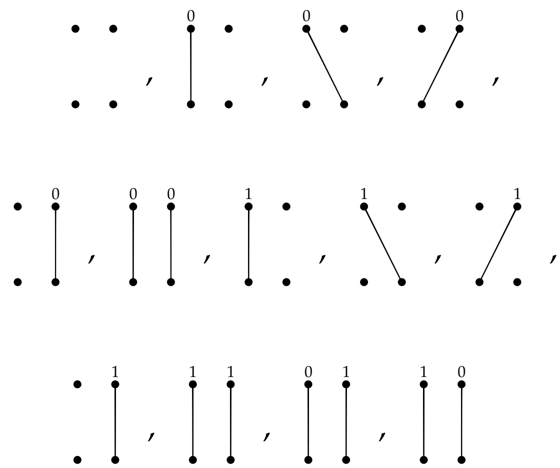


Figure 3.1: The elements of $P_2(\mathbb{Z}_2)$

When we are coloring $d \in P_n$, we are, in a sense, creating functions. A diagram is itself a function $d : \beta(d) \to \tau(d)$, and a coloring c is a function $c : \beta(d) \to G$, where G is our palette group. We could just as easily have defined c to have domain $\tau(d)$. It is important to keep in mind that while the colors are placed above top vertices, they color edges and not vertices.

Theorem 3.1.

The cardinality of $P_n(G)$ is

$$|P_n(G)| = \sum_{k=0}^{n} |G|^k \cdot \binom{n}{k}^2.$$

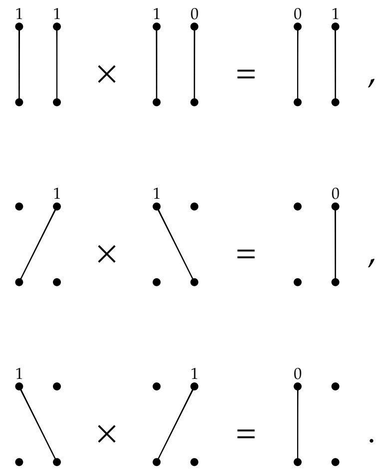
Proof. We know that

$$|P_n| = \sum_{k=0}^{n} \binom{n}{k}^2.$$

For a diagram of rank k, each edge may colored by any element of G, meaning that there are $|G|$ options to color each of the k edges. $\square$

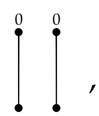
Diagram multiplication now needs to be redefined to incorporate the operation of our palette group, and it is done so in a natural way. We will still stack diagrams.

To determine the color of a newly formed edge, we simply multiply the colors of the component edges. A few examples in $P_2(\mathbb{Z}_2)$ are

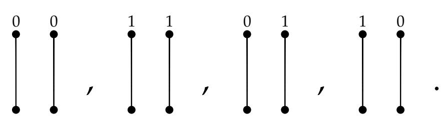


The fact that G is abelian does not imply that multiplication in $P_n(G)$ is commutative, as can be seen in this example. However, whether or not G is abelian does determine some results, and this will be explored Section 3.3.

Consider the group of units of $P_2(\mathbb{Z}_2)$. These are all the elements of $P_2(\mathbb{Z}_2)$ that have multiplicative inverses. The multiplicative identity is



and the units are



We will call this group $P_2(\mathbb{Z}_2)^*$. It should be apparent that $P_2(\mathbb{Z}_2)^* \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In fact, we have the following theorem.

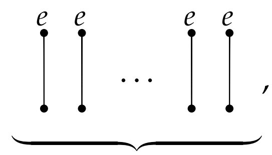
Theorem 3.2.

Let G be a group. Then,

$$P_n(G)^* \cong G^{\times n} = \underbrace{G \times G \times \ldots \times G}_{n \text{ times}},$$

where $P_n(G)^*$ is the group of units of $P_n(G)$.

Proof. The multiplicative identity in $P_n(G)$ is



n times

where e is the identity of G. Recall that for $d_1, d_2 \in P_n$, we have that

$$\text{rank}(d_1 d_2) \leq \min(\text{rank}(d_1), \text{rank}(d_2)).$$

Together with the fact that the multiplicative identity has rank n, we have that the units of $P_n(G)$ must have rank n. Let d be a colored diagram of rank n with coloring c. Then, the inverse of d is the rank n diagram with coloring k such that $k(i) = c(i)^{-1}$. Thus, $d \in P_n(G)$ is a unit if and only if $\text{rank}(d) = n$.

The map φ is clear. Given a diagram d in $P_n(G)^*$ with edges colored $g_1, g_2, \ldots, g_n$ (where the g_i's are not necessarily distinct), d maps to $(g_1, g_2, \ldots, g_n) \in G^{\times n}$.

One-to-one: Let $(g_1, g_2, \ldots, g_n) = (h_1, h_2, \ldots, h_n) \in G^{\times n}$. Then $g_i = h_i$, for all $i \in \{1, 2, \ldots, |S|\}$. The diagrams d and b in $P_n(G)^*$ that are colored $(g_1, g_2, \ldots, g_n)$ and $(h_1, h_2, \ldots, h_n)$, respectively, are then colored in exactly the same way. Thus, $b = d$.

Onto: For $(g_1, g_2, \ldots, g_n) \in G^{\times n}$, we have that the diagram colored $(g_1, g_2, \ldots, g_n)$ maps to it.

Homomorphism: Let $d, b \in P_n(G)^*$ such that d is colored $(g_1, g_2, \ldots, g_n)$ and b is colored $(h_1, h_2, \ldots, h_n)$. Then,

$$\varphi(d)\varphi(b) = (g_1, g_2, \ldots, g_n)(h_1, h_2, \ldots, h_n) = (g_1h_1, g_2h_2, \ldots, g_nh_n) = \varphi(db),$$

as desired.

Thus, φ is an isomorphism, and we have that

$$P_n(G)^* \cong G^{\times n} = \underbrace{G \times G \times \ldots \times G}_{n \text{ times}}.$$

□

We now wish to generalize further. Define

$$P_n(G)[S] = \{d \in P_n \mid \tau(d) = \beta(d) = S \subseteq \{1, \ldots, n\}\}.$$

The elements of $P_n(G)[S]$ are simply the diagrams in $P_n(G)$ with top and bottom sets both S. That is, they look like subdiagrams of unit elements of $P_n(G)$.

Example 3.1. The group $P_3(\mathbb{Z}_3)[\{1,3\}]$ consists of the elements in Figure 3.2.

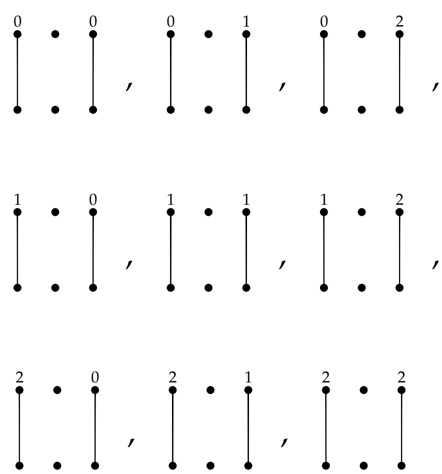


Figure 3.2: The elements of $P_3(\mathbb{Z}_3)[\{1,3\}]$

Note that $P_n(G)[\{1,\ldots,n\}]$ is just $P_n(G)^*$. In general, we have the following theorem.

Theorem 3.3.

Let G be a group. Then,

$$P_n(G)[S] \cong G^{\times|S|} = \underbrace{G \times G \times \ldots \times G}_{|S|\ \textit{times}}.$$

Proof. The map φ is the one described in Theorem 3.2. Given a diagram d in $P_n(G)[S]$ with edges colored $g_1, g_2, \ldots, g_{|S|}$ (where the g_i's are not necessarily distinct), d maps to $(g_1, g_2, \ldots, g_{|S|}) \in G^{\times|S|}$.

One-to-one: Let $(g_1, g_2, \ldots, g_{|S|}) = (h_1, h_2, \ldots, h_{|S|}) \in G^{\times|S|}$. Then $g_i = h_i$, for all $i \in \{1, 2, \ldots, |S|\}$. The diagrams d and b in $P_n(G)[S]$ that are colored $(g_1, g_2, \ldots, g_{|S|})$ and $(h_1, h_2, \ldots, h_{|S|})$, respectively, are then colored in exactly the same way. Thus, $b = d$.

Onto: For $(g_1, g_2, \ldots, g_{|S|}) \in G^{\times|S|}$, we have that the diagram colored $(g_1, g_2, \ldots, g_{|S|})$ maps to it.

Homomorphism: Let $d, b \in P_n(G)[S]$ such that d is colored $(g_1, g_2, \ldots, g_{|S|})$ and b is colored $(h_1, h_2, \ldots, h_{|S|})$. Then,

$$\varphi(d)\varphi(b) = (g_1, g_2, \ldots, g_{|S|})(h_1, h_2, \ldots, h_{|S|}) = (g_1h_1, g_2h_2, \ldots, g_{|S|}h_{|S|}) = \varphi(db),$$

as desired.

Thus, φ is an isomorphism, and we have that

$$P_n(G)[S] \cong G^{\times|S|} = \underbrace{G \times G \times \ldots \times G}_{|S|\ \text{times}}.$$

$\square$

Inside $P_n(G)$, we have found $\binom{n}{k}$ copies of $G^{\times k}$ for $k \in \{1, \ldots, n\}$, one for each choice of rank k diagram with equal top and bottom sets of vertices.

Recall (from the end of Chapter 1) that we have information on the semisimplicity of group algebras. Given our groups $P_n(G)[S]$, we can apply Theorem 1.7.

Theorem 3.4.

The complex span of $P_n(G)[S]$, denoted $\mathbb{C}P_n(G)[S]$, is semisimple.

Proof. By Theorem 3.3, $P_n(G)[S] \cong G^{\times|S|}$. Thus, $\mathbb{C}P_n(G)[S] \cong \mathbb{C}G^{\times|S|}$ is a group algebra (Example 1.5). Of course, $\mathbb{C}$ has characteristic zero. By Theorem 1.7, $\mathbb{C}P_n(G)[S]$ is semisimple. □

We have thus found a collection of semisimple subalgebras of the colored planar rook algebra $\mathbb{C}P_n(G)$. However, the groups $P_n(G)[S]$ are only a small part of $\mathbb{C}P_n(G)$. In the next section, we look at the potential semisimplicity of $\mathbb{C}P_n(G)$ itself.

3.2 $\mathbb{C}P_n(G)$ and Semisimplicity: A Mirrored Approach

In decomposing $\mathbb{C}P_n(G)$, we are looking to mirror the approach to decomposing the uncolored version of $\mathbb{C}P_n$ (Chapter 2). This section is devoted to what we believe to be the closest mirror and why an approach too similar to that which we used with uncolored $\mathbb{C}P_n$ does not work.

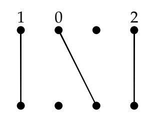
We start by mirroring Section 2.2, "The Action on P_n." We want to look at how $P_n(G)$ acts on a set, as P_n did in Definition 2.1. We can still define $\tau(d)$ and $\beta(d)$ for a diagram $d \in P_n(G)$. However, these no longer uniquely determine the diagram, as there are $|G|^k$ possible colorings of d, where $k = \text{rank}(d)$.

Now, a planar rook diagram d is not an injection from $\beta(d)$ to $\tau(d)$, considering that there is no longer a bijection between diagrams and pairs $(\beta(d), \tau(d))$, where $|\beta(d)| = |\tau(d)|$. However, taking $\tau(d)$ and $\beta(d)$ together with a color scheme $(g_1, g_2, \ldots, g_k)$, where g_i is the color of the ith left-most edge of d, we have that d is uniquely determined. This is somewhat cumbersome, so to simplify we extend the set from which we are taking subsets on which d may act. Previously, we let $S \subseteq \{1, \ldots, n\}$. Now, working in $P_n(G)$, define

$$S(G)_n = \{1_i, 2_i, \ldots, n_i \mid i \in G\}.$$

That is, we are looking at subsets of the set that consists of $|G|$ copies of 1 through n, each indexed by a different element of our palette group.

Now, $\tau(d)$ and $\beta(d)$ as equal sized subsets of $S(G)_n$ completely and uniquely determine $d \in P_n(G)$. For example, when working in $P_4(\mathbb{Z}_4)$, letting $\tau(d) = \{1_3, 2_1, 4_1\}$ and $\beta(d) = \{1_2, 3_1, 4_3\}$, we have completely and uniquely determined d as

$d =$ 

.

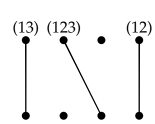
Recall that the colors on d are edge colors. The way they are determined is by looking at what element of $\mathbb{Z}_4$ maps the subscript on the vertex labeled v_i in the bottom to the subscript on the vertex labeled $d(v)_j$ in the top. For example, in $\beta(d)$, the index on 1 is 2 and in $\tau(d)$, the index on 1 is 3. Then, the diagram needs to map 2 to 3. Thus, we color the edge from 1 to 1 by 1, as, in $\mathbb{Z}_4$, $2 + 1 = 3$.

Let $X \subseteq \{1, \ldots, n\}$ and $H \subseteq G$. Then, if $S \subseteq \left\{s_j \mid s \in X, j \in H\right\} \subseteq S(G)_n$, we have that

$$d(S) = \{d(s)_\ell\},$$

where $\ell = gj$, in which g is the color of the edge incident to s in d.

Example 3.2. We will work in $P_4(S_3)$. Let $S = \left\{1_{(12)}, 3_{(132)}\right\}$. Let

$d =$ 

.

Denote by S^* the set S with indices of elements ignored. Here, for example,

$S^* = X = \{1, 3\}$. Certainly, $S^* \subseteq \beta(d)$. Then, $d(S) = \left\{1_{(123)}, 2_e\right\}$, as (13)(12) = (123) and (123)(132) = e (recall that we are multiplying right to left).

As in Section 2.2, we look to a vector space. Let $V^n(G)$ be the $2^{(n\times|G|)}$-dimensional complex vector space with basis

$$\{\vec{v}_S \mid S \subseteq S(G)_n\}.$$

Now, $d \in P_n(G)$ acts on the basis elements of $V^n(G)$ by

$$(\vec{v}_S)^d = \begin{cases} \vec{v}_{d(S)}, & \text{if } S \subseteq \beta(d), \\ \vec{0}, & \text{otherwise.} \end{cases}$$

If we restrict $|S|$ to a certain size, say k, we get a P_n-invariant subspace of $V^n(G)$. Call this subspace

$$V_k^n(G) = \mathbb{C}\text{-span}\,\{\vec{v}_S \mid S \subseteq S(G)_n \text{ and } |S| = k\}.$$

When uncolored, this was an irreducible subspace of V^n. Now, we have to check whether $V_k^n(G)$ is irreducible. We will not do this just yet, but keep in mind that we do not know whether $V_k^n(G)$ is irreducible for the moment.

Next, we define the colored planar rook algebra by

$$\mathbb{C}P_n(G) = \mathbb{C}\text{-span}\,\{d \mid d \in P_n(G)\}.$$

Let us see if we can change to an x_d-style basis. Define x_d as in Equation 2.2. Now, if $b \subseteq d$, then any edge in b colored j must be colored j in d, as well. The x_d are linearly independent, and we can form linear combinations of x_d to get $d \in P_n(G)$, as before. That is, the set of x_d forms a basis for $\mathbb{C}P_n(G)$.

The action of $b \in P_n(G)$ on $x_d \in \mathbb{C}P_n(G)$ is then

$$(x_d)^b = \begin{cases} x_{bd}, & \text{if } \tau(d) \subseteq \beta(b), \\ \vec{0}, & \text{otherwise.} \end{cases}$$

Rewrite the x_d's as in Section 2.3, but including the coloring, as $x_{S,T,(g_1,\ldots,g_k)}$.

Consider

$$W_T^{n,k} = \mathbb{C}\text{-span}\left\{x_d \mid \operatorname{rank}(d) = k, \beta(d) = T\right\}.$$

We want to show that $W_T^{n,k} \cong V_k^n(G)$. Let us try a map similar to what we saw in Section 2.3:

$$x_{S,T,(g_1,\ldots,g_k)} \mapsto \vec{v}_S,$$

where the S in $\vec{v}_S$ is the S in $x_{S,T}$ colored by $(g_1, \ldots, g_k)$.

We first check to see if the map is a bijection. Let $\vec{v}_{S_1} = \vec{v}_{S_2}$. Then, $S_1 = S_2$. As this determines the coloring of $x_{S,T}$, we see that $x_{S_1,T} = x_{S_2,T}$. This gives us injectivity. Let $\vec{v}_S \in V_k^n(G)$. Then, $x_{S,T,C}$, where C is the coloring determined by the subscripts on the values of S in $\vec{v}_S$, maps to $\vec{v}_S$. This gives us surjectivity. Thus, we have a bijection.

Again, due to linearity, we have a homomorphism. Hence,

$$W_T^{n,k} \cong V_k^n(G),$$

as desired.

Theorem 2.2 told us that V_k^n (and thus $W_T^{n,k}$) was irreducible when we were concerned only with $\{1, 2, \ldots, n\}$ and P_n, uncolored. When we incorporate colors, however, $W_T^{n,k}$ is not necessarily irreducible.

Theorem 3.5.

Let G be a group with $|G| = m > 1$. Then, $W^{1,1}_{\{1\}}$ is a $P_1(G)$-invariant subspace of $\mathbb{C}P_1(G)$ that is not irreducible.

Proof. Since $W^{1,1}_{\{1\}} \cong V^1_1(G)$, we can show that $V^1_1(G)$ is not irreducible.

Note that the indices on the following vectors are singleton sets, but for the sake of notation, we do not write the set brackets.

Consider

$$U := \langle \vec{v}_{1_{g_1}} + \vec{v}_{1_{g_2}} + \ldots + \vec{v}_{1_{g_m}} \rangle \subseteq V^1_1(G),$$

where $g_i = g_j$ if and only if $i = j$.

We first show that U is $P_1(G)$-invariant. As the left-regular action of G on itself is indeed regular, we see that $\vec{v}_{1_{g_1}} + \vec{v}_{1_{g_2}} + \ldots + \vec{v}_{1_{g_m}}$ under the action of $d \in P_1(G)$ is either equal to the zero vector (if d is the empty diagram) or permutes the subscripts in $\vec{v}_{1_{g_1}} + \vec{v}_{1_{g_2}} + \ldots + \vec{v}_{1_{g_m}}$. In the latter case, $\vec{v}_{1_{g_1}} + \vec{v}_{1_{g_2}} + \ldots + \vec{v}_{1_{g_m}}$ under d is equal to itself, as vector addition is commutative. Thus, U is $P_1(G)$-invariant.

Certainly, U is a subspace of $W^{1,1}_{\{1\}}$, as it only consists of scalar multiples of

$$\vec{v}_{1_{g_1}} + \vec{v}_{1_{g_2}} + \ldots + \vec{v}_{1_{g_m}}.$$

It is also clear that U is a proper subspace of $W^{1,1}_{\{1\}}$, as it does not contain, $\vec{v}_{1_{g_1}}$, for example.

Hence, $W^{1,1}_{\{1\}}$ is not irreducible. $\square$

Note that $W^{1,1}_{\{1\}}$ is isomorphic as an algebra to the group algebra $\mathbb{C}P_1(G)[\{1\}] \cong \mathbb{C}G$. Thus, by Theorem 1.7, $W^{1,1}_1$ is semisimple. This seems to imply that we have some hope of our mirrored approach working, we just have to break down our subspaces even further. However, we now generalize Theorem 3.5, showing that $W^{1,1}_{\{1\}}$ is not the only problem subspace.

Theorem 3.6.

Let G be a group with $|G| > n$. Then, $W^{n,n}_{\{1,2,\ldots,n\}}$ is a $P_n(G)$-invariant subspace of $\mathbb{C}P_n(G)$ that is not irreducible.

Proof. For the sake of brevity, let $A = \{1, 2, \ldots, n\}$. Since $W^{n,n}_A \cong V^n_n(G)$, we can show that $V^n_n(G)$ is not irreducible.

Recall that S^* is the set S with indices of elements ignored. Consider

$$U := \left\langle \sum_{S \subseteq S(G)_n,\ S^*=A} \vec{v}_S \right\rangle \subseteq V^n_n(G).$$

We first show that U is $P_n(G)$-invariant. As the left-regular action of G on itself is indeed regular, we see that

$$\sum_{S \subseteq S(G)_n,\ S^*=A} \vec{v}_S$$

under the action of $d \in P_n(G)$ is either equal to the zero vector (if $\mathrm{rank}(d) < n$) or permutes the subscripts in

$$\sum_{S \subseteq S(G)_n,\ S^*=A} \vec{v}_S.$$

In the latter case,

$$\left(\sum_{S \subseteq S(G)_n,\ S^*=A} \vec{v}_S \right)^d = \sum_{S \subseteq S(G)_n,\ S^*=A} \vec{v}_S,$$

as vector addition is commutative. Thus, U is $P_n(G)$-invariant.

Certainly, U is a subspace of $W^{n,n}_A$, as it only consists of scalar multiples of

$$\sum_{S \subseteq S(G)_n,\ S^*=A} \vec{v}_S.$$

It is also clear that U is a proper subspace of $W^{n,n}_A$. Let $T \subseteq S(G)_n$ with $|T| = n$. Then, U does not contain $\vec{v}_T$.

Hence, $W^{n,n}_A$ is not irreducible. $\square$

Our mirrored approach to decomposing $\mathbb{C}P_n(G)$ has clearly not gone very well. Subspaces that were irreducible in $\mathbb{C}P_n$ become non-irreducible when colored, meaning that the decomposition process (or lack thereof) is significantly different. We must turn to a new, more complicated method.

3.3 $\mathbb{C}P_n(G)$ and Semisimplicity: A New Approach

It is clear that when we introduce colors, the path to semisimplicity (if one exists) becomes harder to navigate. To decompose $\mathbb{C}P_n(G)$, with G finite abelian, we need a mixture of notation, cleverly-defined subspaces, and geometric intuition.

3.3.1 The Fundamental Theorem of Finite Abelian Groups

Theorem 3.7 (The Fundamental Theorem of Finite Abelian Groups)**.**
Let G be a finite abelian group. Then, G is isomorphic to the direct product of cyclic groups.

Most undergraduate proofs of Theorem 3.7 require knowledge of the Sylow Theorems, which are not covered in this Independent Study. Thus, we leave Theorem 3.7 without proof. The Sylow Theorems are concerned with Sylow subgroups of a given finite group G. Sylow subgroups are related to the prime decomposition of the order of G. The Sylow Theorems are some of the most interesting and powerful group-theoretic notions (that may be encountered by an undergraduate) and alone could make for an entire Independent Study. However, as we are not going into detail about them here, we are not covering a proof of The Fundamental Theorem of Finite Abelian Groups.

Example 3.3. Let G be an abelian group of order 63. We claim that there are 2 such groups, up to isomorphism:

1. $\mathbb{Z}_9 \times \mathbb{Z}_7$,

2. $\mathbb{Z}_7 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

One may be tempted to question why $\mathbb{Z}_{63}$ was not listed in Example 3.3. This is because $\mathbb{Z}_{63} \cong \mathbb{Z}_9 \times \mathbb{Z}_7$. In general, if $\gcd(a, b) = 1$, we have that $\mathbb{Z}_{ab} \cong \mathbb{Z}_a \times \mathbb{Z}_b$. This is also why $\mathbb{Z}_{21} \times \mathbb{Z}_3$ is not listed, as it is isomorphic to $\mathbb{Z}_7 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, considering $\gcd(7, 3) = 1$.

3.3.2 $\mathbb{C}P_n(G)$ For Finite Abelian G

In this section, we will consider finite abelian groups as the direct products of cyclic groups (see Theorem 3.7). That is, we will think of a finite abelian group G as

$$G = \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \ldots \times \mathbb{Z}_{q_m},$$

where $q_i \in \mathbb{N}$. Then, elements of G look like

$$g = (g_1, g_2, \ldots, g_m),$$

where $0 \leq g_i < q_i$.

Definition 3.1 Let G be a group. A G**-partition** of $\{1, 2, \ldots, n\}$ is a set

$$A = \left\{A_g \mid g \in G\right\}$$

of pairwise-disjoint subsets A_g of $\{1, 2, \ldots, n\}$ indexed by elements of G such that

$$\bigcup_{g \in G} A_g \subseteq \{1, 2, \ldots, n\}.$$

That is, while we use the term "partition," we are only looking at a partition of a subset of $\{1, 2, \ldots, n\}$ [2].

Example 3.4. First, we need a group. Let us work with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. We also need to determine a value for n. That is, we are working with diagrams in P_n for some n. Let $n = 7$. Then,

$$A = \{\{1, 2\}, \{5\}, \{6\}, \{7\}\}$$

is a collections of subsets of $\{1, 2, 3, 4, 5, 6, 7\}$. Let $A_{(0,0)} = \{1, 2\}$, $A_{(0,1)} = \{5\}$, $A_{(1,0)} = \{6\}$, and $A_{(1,1)} = \{7\}$. Now, A is a G-partition of $\{1, 2, 3, 4, 5, 6, 7\}$.

Definition 3.2 Let A be a G-partition of $\{1, 2, \ldots, n\}$, S the bottom of a diagram d with $\bigcup_{g \in G} A_g \subseteq S$, and $c : S \to G$ a color scheme. Define the α**-function**

$$\alpha(A, c) = \prod_{g \in G} \prod_{i \in A_g} \prod_{1 \leq j \leq m} \left(\zeta_{q_j}\right)^{g_j c(i)_j},$$

where ζ_{q_j} is the q_jth root of unity $e^{\frac{2\pi i}{q_j}}$. Note that g_j is the jth entry in the tuple g. Similarly, $c(i)_j$ is the jth entry of the tuple that is the color on edge i of d. We define the empty product as being equal to 1 [2].

The α-function is a rule for extracting a root of unity out of a partition/coloring pair. Despite the triple-product, when the group is fixed, the partition determines the value of the α-function. To understand this product, we will work through an example.

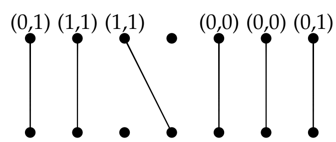
Example 3.5. We will continue with the group and G-partition from Example 3.4.

We can now pick an S such that $\bigcup_{g \in G} A_g = \{1, 2, 5, 6, 7\} \subseteq S$. Let $S = \{1, 2, 4, 5, 6, 7\}$.

Let $c : S \to G$ be the color scheme defined by

$$\begin{aligned} 1 &\mapsto (0,1) \\ 2 &\mapsto (1,1) \\ 4 &\mapsto (1,1) \\ 5 &\mapsto (0,0) \\ 6 &\mapsto (0,0) \\ 7 &\mapsto (0,1). \end{aligned}$$

One diagram (there are multiple) that represents this color scheme is



.

We can now find $\alpha(A, c)$. We will do so in pieces, as the task is not an easy one. Fix $g = (0,0)$. Then, we consider

$$\prod_{i \in A_{(0,0)}} \prod_{1 \le j \le 2} \left(\zeta_{q_j}\right)^{(0,0)_j c(i)_j}.$$

Now, $A_{(0,0)} = \{1, 2\}$. Letting $i = 1$, we have

$$\begin{aligned} \left(\zeta_{q_1}\right)^{(0,0)_1 c(1)_1} \left(\zeta_{q_2}\right)^{(0,0)_2 c(1)_2} &= (\zeta_2)^{0 \cdot c(1)_1} (\zeta_2)^{0 \cdot c(1)_2} \\ &= (\zeta_2)^{0 \cdot (0,1)_1} (\zeta_2)^{0 \cdot (0,1)_2} \\ &= (\zeta_2)^{0 \cdot 0} (\zeta_2)^{0 \cdot 1} \\ &= 1. \end{aligned}$$

Letting $i = 2$, we have

$$\begin{aligned}
\left(\zeta_{q_1}\right)^{(0,0)_1 c(2)_1} \left(\zeta_{q_2}\right)^{(0,0)_2 c(2)_2} &= (\zeta_2)^{0\cdot c(2)_1} (\zeta_2)^{0\cdot c(2)_2} \\
&= (\zeta_2)^{0\cdot(1,1)_1} (\zeta_2)^{0\cdot(1,1)_2} \\
&= (\zeta_2)^{0\cdot 1} (\zeta_2)^{0\cdot 1} \\
&= 1.
\end{aligned}$$

Thus,

$$\prod_{i\in A_{(0,0)}} \prod_{1\leq j\leq 2} \left(\zeta_{q_j}\right)^{(0,0)_j c(i)_j} = 1.$$

Continuing,

$$\begin{aligned}
\prod_{i\in A_{(0,1)}} \prod_{1\leq j\leq 2} \left(\zeta_{q_j}\right)^{(0,1)_j c(5)_j} &= (\zeta_2)^{0\cdot c(5)_1} (\zeta_2)^{1\cdot c(5)_2} = (\zeta_2)^{0\cdot 0} (\zeta_2)^{1\cdot 0} = 1, \\
\prod_{i\in A_{(1,0)}} \prod_{1\leq j\leq 2} \left(\zeta_{q_j}\right)^{(1,0)_j c(6)_j} &= (\zeta_2)^{1\cdot c(6)_1} (\zeta_2)^{0\cdot c(6)_2} = (\zeta_2)^{1\cdot 0} (\zeta_2)^{0\cdot 0} = 1, \\
\prod_{i\in A_{(1,1)}} \prod_{1\leq j\leq 2} \left(\zeta_{q_j}\right)^{(1,1)_j c(7)_j} &= (\zeta_2)^{1\cdot c(7)_1} (\zeta_2)^{1\cdot c(7)_2} = (\zeta_2)^{1\cdot 0} (\zeta_2)^{1\cdot 1} = \zeta_2 = e^{\pi i} = -1.
\end{aligned}$$

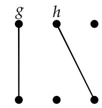
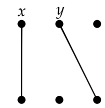
Thus, $\alpha(A, c) = 1 \times 1 \times 1 \times -1 = -1$.

When A is a G-partition of $\{1, 2, \ldots, n\}$ and $d \in P_n(G)$, where $d : S \to T$ and $\bigcup_{g\in G} A_g \subseteq S$, we can look at

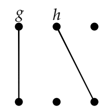
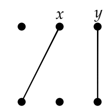
$$d(A) = \left\{ d(A_g) \mid g \in G \right\}.$$

Note that $d(A)$ is a G-partition of $\{1, 2, \ldots, n\}$. The block indexed by $g \in G$ is $d(A_g)$.

To denote that a diagram d has color scheme c, we write d_c. When colors are ignored, d_c and d_k are the same diagram. For example, it may be that

$d_c =$  and $d_k =$  ,

but it is never the case that, say,

$d_c =$  and $d_k =$  ,

because the underlying diagrams are different. When discussing $\beta(d)$ and $\tau(d)$, we need not specify a color scheme, as $\beta(d_c) = \beta(d_k)$ for all c and k (the same goes for $\tau(d)$).

Lemma 3.8. *Let $d_c, b_k \in P_n(G)$ with $\tau(d) \subseteq \beta(b)$.*

1. *If $b_k d_c = (bd)_r$, then for any G-partition of $\{1, 2, \ldots, n\}$, we have that*

$$\alpha(A, r) = \alpha(d(A), k) \cdot \alpha(A, c);$$

2. *If A_1 and A_2 are G-partitions of $\{1, 2, \ldots, n\}$ with $\bigcup_{g\in G} A_{1,g} \cap \bigcup_{g\in G} A_{2,g} = \emptyset$ and $\bigcup_{g\in G} A_{1,g} \subseteq \beta(d)$ and $\bigcup_{g\in G} A_{2,g} \subseteq \beta(d)$, then*

$$\alpha(A_1 \cup A_2, c) = \alpha(A_1, c) \cdot \alpha(A_2, c),$$

where $A_1 \cup A_2$ is the G-partition of $\{1, 2, \ldots, n\}$ where the block indexed by $g \in G$ is $A_{1,g} \cup A_{2,g}$ [2].

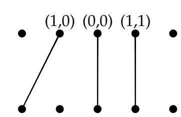
While the notation is dense, the underlying concept is straightforward. Lemma 3.8 (1) says that given a partition, the α-function should be multiplicative – it should return the same value when applied to the product of diagrams as it does when applied to the diagrams individually and those results are multiplied together.

Two diagrams d and b can be said to be disjoint if $\beta(d) \cap \beta(b) = \tau(d) \cap \tau(b) = \emptyset$. Equivalently, if the associated matrices M_d and M_b have the property that the (i, j)-entry of M_d times the (k, ℓ)-entry of M_b is zero for all i, j, k, ℓ and $\beta(d) \cap \beta(b) = \emptyset$. Lemma 3.8 (2) says that given two disjoint diagrams, we can effectively impose one on top of the other. The α-function should return the same value when applied to this "imposition" as it does when applied to the diagrams individually and those results are multiplied together.

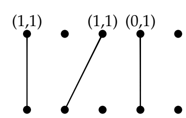
Lemma 3.8 holds in large part due to the fact that we are working with finite abelian groups – we could not even define the α-function otherwise.

We now exemplify part (1) of Lemma 3.8.

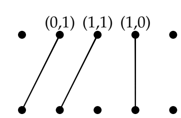
Example 3.6. We will work in $\mathbb{Z}_2 \times \mathbb{Z}_2$, with

$b_k =$ 

,

$d_c =$ 

,

$b_k d_c = (bd)_r =$ 

.

Let our partition be $A = \left\{\{1,2\}_{(0,0)}, \{4\}_{(0,1)}\right\}$. Then, $d(A) = \left\{\{1,3\}_{(0,0)}, \{4\}_{(0,1)}\right\}$.

We can now find $\alpha(A,c)$, $\alpha(d(A),k)$, and $\alpha(A,r)$:

$$\alpha(A,c) = \left(\zeta_2^0\right)^5 \times \zeta_2 = -1,$$

$$\alpha(d(A),c) = \left(\zeta_2^0\right)^3 \times \zeta_2 = -1,$$

$$\alpha(A,r) = \left(\zeta_2^0\right)^4 = 1.$$

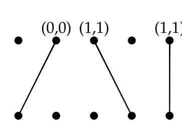
As desired,

$$\alpha(d(A),k) \cdot \alpha(A,c) = -1 \times -1 = 1 = \alpha(A,r).$$

Part (1) of Lemma 3.8 discusses the multiplicativeness of the α-function. Furthermore, it helps us see how similar to a homomorphism the α-function is. We are taking colored diagrams and sending them through the α-function to points on the unit circle. These points depend upon the number of cyclic groups in the decomposition of our chosen group (that is, the points depend upon m in $\mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_m}$). Then, we are sending our sets of diagrams to subgroups of the group of rotations of the unit circle. In fact, we appear to be sending them to $\mathbb{Z}_m$.

Let us now look at an example of part (2) of Lemma 3.8.

Example 3.7. We again work in $\mathbb{Z}_2 \times \mathbb{Z}_2$. We will look at the partitions $A_1 = \left\{\{1\}_{(0,0)}\right\}$ and $A_2 = \left\{\{4\}_{(0,0)}, \{5\}_{(0,1)}\right\}$. Then, $A_1 \cup A_2 = \left\{\{1,4\}_{(0,0)}, \{5\}_{(0,1)}\right\}$. Let

$d_c =$  .

Calculation yields that $\alpha(A_1,c) = 1$, $\alpha(A_2,c) = -1$, and $\alpha(A_1 \cup A_2, c) = -1$.

Note that the product

$$\prod_{i \in A_g} \prod_{1 \le j \le m} \left(\zeta_{q_j}\right)^{g_j c(i)_j} = 1, \tag{3.1}$$

when $g = e$, which helps to solidify the idea that the α-function behaves like a homomorphism into $\mathbb{Z}_m$.

In Chapter 2, we defined a partial order on P_n via rank and extended this to a total order. However, our examples in Section 3.2 show us that this will not work for decomposing general $\mathbb{C}P_n(G)$. To achieve our desired decomposition, we define a different partial ordering, this time in terms of G-partitions. Given G-partitions A_1 and A_2, we define a partial order by $A_1 \leq A_2$ if and only if $A_{1,g} = A_{2,g}$ for all $g \neq e$ and $A_{1,e} \subseteq A_{2,e}$.

Example 3.8. Figure 3.3 shows an example of the partial ordering: two partitions A_1 and A_2 have the relation $A_1 \leq A_2$ if there is a path of arrows from A_1 to A_2. In the figure, let $g \neq e$.

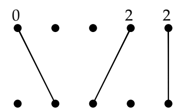
$$\begin{tikzcd}
 & & & \{\{1,2\}_e, \{3\}_g\} & & & \\
\{\{1\}_e, \{3\}_g\} \arrow[urrr] & & \{\{2\}_e, \{3\}_g\} \arrow[ur] & & \{\{4\}_e, \{3\}_g\} & & \{\{4\}_e, \{2\}_g\} \\
 & & & \{\{3\}_g\} \arrow[ulll] \arrow[ul] \arrow[ur] & & &
\end{tikzcd}$$

Figure 3.3: An example of the partial ordering on G-partitions

In terms of the α-function, Equation 3.1 implies that $\alpha(A, c) = \alpha(B, c)$ for all $B \leq A$.

At this point, we have seen some similarities between this approach to decomposing $\mathbb{C}P_n(G)$ and the approach we took to decomposing $\mathbb{C}P_n$, uncolored. The partitions we are working with are analogous to the T's in the $W_T^{n,k}$'s (Equation 2.4).

This will become more apparent later, but consider how partitions correspond to diagrams. Let us look at

$d_c =$  $\in P_5(\mathbb{Z}_3).$

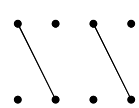
Specifically, we want to consider $\beta(d)$ and the color scheme c. Recall that $W_T^{n,k} = \mathbb{C}\text{-span}\,\{x_d \mid \text{rank}(d) = k, \beta(d) = T\}$. We fixed a bottom set and ran through possible top sets to create this subspace, and this is essentially what we are doing with the partitions. The partition that corresponds to d_c is $\{\{2\}_0, \{3, 5\}_2\}$. This partition corresponds to all diagrams in $P_5(\mathbb{Z}_3)$ with bottom set $\{2, 3, 5\}$ such that the color of the edge incident to 2 is 0 and the color of the edges incident to 3 and 5 is 2.

We can extend to a total ordering as before, preserving the partial ordering. We now define the objects that are analogous to the x_d's of the uncolored planar rook algebra. Let A be a G-partition of $\{1, 2, \ldots, n\}$, and let $T \subseteq \{1, 2, \ldots, n\}$ with $|T| = \left|\bigcup_{g\in G} A_g\right|$. Define

$$y_A^T = \sum_{B\leq A}\left((-|G|)^{|A_0/B_0|}\left(\sum_{c\in \text{Col}(d|_{\cup B})} \alpha(B, c)(d|_{\cup B})_c\right)\right), \tag{3.2}$$

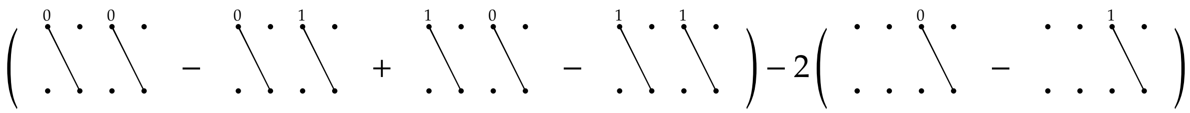
where $|A_0/B_0|$ is $|A_0| - |B_0|$, d is the diagram with $\beta(d) = \bigcup_{g\in G} A_g$ and $\tau(d) = T$, $\text{Col}(d)$ is the set of possible color schemes of d, and $d|_{\cup B}$ is the diagram d with $\beta(d)$ restricted to $\bigcup_{g\in G} B_g$. Indeed, $d|_{\cup B}$ is a subdiagram of d. The subscript c on $d|_{\cup B}$ at the end of Equation 3.2 indicates how the diagram $d|_{\cup B}$ is colored.

Example 3.9. We will work in $\mathbb{Z}_2$, with $A = \{\{2\}_0, \{4\}_1\}$. Let $T = \{1, 3\}$. Then,

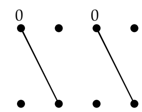
$d =$  .

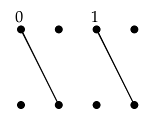
We leave d uncolored, as we will be looking at all possible colorings. Then,

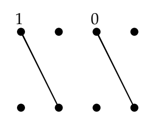
$y_A^T = y^{\{1,3\}}_{\{\{2\}_0,\{4\}_1\}}$ is equal to

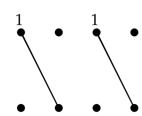


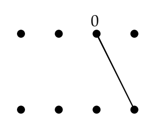
This comes from the definition of y_A^T and the following calculations:

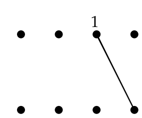
$\alpha\big(\{\{2\}_0, \{4\}_1\},$  $\big) = 1,$

$\alpha\big(\{\{2\}_0, \{4\}_1\},$  $\big) = -1,$

$\alpha\big(\{\{2\}_0, \{4\}_1\},$  $\big) = 1,$

$\alpha\big(\{\{2\}_0, \{4\}_1\},$  $\big) = -1,$

$\alpha\big(\{\{4\}_1\},$  $\big) = 1,$

$\alpha\big(\{\{4\}_1\},$  $\big) = -1.$

Here, the diagrams represent color schemes.

If we look at $\mathbb{C}P_n(\{e\})$, only coloring with the trivial group, $y_A^T = x_{T,A_0}$. Recall that x_{T,A_0} is the x_d with $\tau(d) = T$ and $\beta(d) = A_0$. This equality occurs due to the fact that subpartitions of A correspond to subsets of A_0 and because the α-function does not contribute to y_A^T, since we are coloring by e.

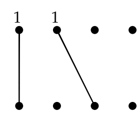
The action of $P_n(G)$ on the y_A^T's is as expected.

Theorem 3.9 ([2]).

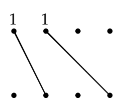
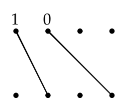
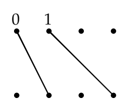
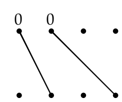
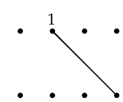
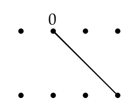
For $d_c, b_k \in P_n(G)$ with A a G-partition of $\{1, \ldots, n\}$ such that $d_c : \bigcup_{g \in G} A_g \to T$, we have that

$$\left(y_A^T\right)^{b_k} = \begin{cases} \alpha(d(A), k)^{-1} \cdot y_A^{b(T)}, & \text{if } T \subseteq \beta(b), \\ \vec{0}, & \text{otherwise.} \end{cases}$$

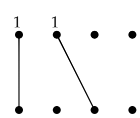
Example 3.10. Working in $\mathbb{Z}_2$, consider $y_A^T = y_{\{\{2\}_0, \{4\}_1\}}^{\{1,3\}}$ from Example 3.9, and let

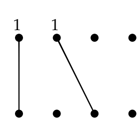
$b_k =$  .

Then, $\left(y_{\{\{2\}_0, \{4\}_1\}}^{\{1,3\}}\right)^{b_k}$ is equal to

$\Big($ $-$  $+$  $-$  $+$  $\Big)$ $-2\Big($ $-$  $+$  $\Big)$,

which involves the following calculations:

$\alpha\Big(\{\{1\}_0, \{2\}_1\},$  $\Big) = -1,$

$\alpha\Big(\{\{2\}_1\},$  $\Big) = -1.$

Here, the diagrams represent color schemes.

Note what the action of $P_n(G)$ on the y_A^T's reveals about the α-function. The α-function creates an invariant-like complex number that is associated to a diagram. When a diagram acts on y_A^T, we need to update the associated scalars to fit the new diagrams.

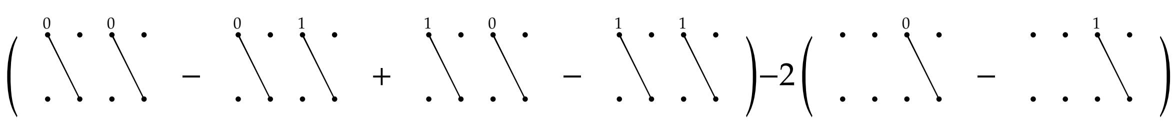
Recall that when we encountered the x_d's, we rewrote them in terms of each other. This occurred in the proof of Theorem 2.3. We can do something similar with the y_A^T's.

Lemma 3.10. *We can write the y_A^T's in terms of each other as follows:*

$$y_A^T = \sum_{c \in Col(d)} \alpha(A,c)d_c - \left(\sum_{B<A} \left((|G|)^{|A_0/B_0|} y_B^{d(\cup_{g\in G} B_g)} \right) \right).$$

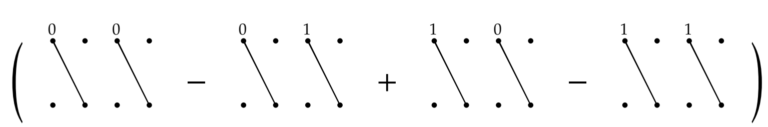
Proof. Note that the second sum runs over proper preceding partitions. That is, we are looking at $B < A$, rather than at $B \leq A$. When $B = A$, we have that $|A_0/B_0| = 0$, meaning that $-|G|^{|A_0/B_0|} = 1$. Thus, the first term in a given y_A^T depends only upon the sum over the possible coloring schemes. The result follows. □

Example 3.11. Continuing from our previous examples, let us work in $\mathbb{C}P_4(\mathbb{Z}_2)$, and consider $y^{\{1,3\}}_{\{\{2\}_0,\{4\}_1\}}$:



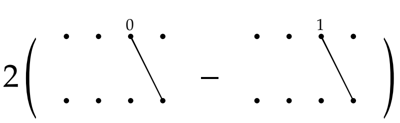
Lemma 3.10 tells us that we can consider $y^{\{1,3\}}_{\{\{2\}_0,\{4\}_1\}}$ in terms of y_B^T's that satisfy certain conditions. We first need to know which partitions B satisfy $B \leq A$. There exist two such partitions: A and $\{\{4\}_1\}$. Now, we can see that

$$\sum_{c\in \mathrm{Col}(d)} \alpha(A,c)d_c =$$



There is only one proper subpartition of A, so

$$\sum_{B<A} \left((|G|)^{|A_0/B_0|} y_B^{d(\cup_{g\in G} B_g)} \right) = (|G|)^{|A_0/B_0|} y_B^{d(\cup_{g\in G} B_g)} = 2$$



The difference of these two summations is indeed $y^{\{1,3\}}_{\{\{2\}_0,\{4\}_1\}}$.

Our next big claim will be that the y_A^T's form a basis for $\mathbb{C}P_n(G)$. To prove this, we need the following lemma.

Lemma 3.11. *Let $d \in P_n(G)$ with $d : S \to T$, and let* $\operatorname{rank}(d) = k$. *For any colorings c and k of d, we have that*

$$\sum_{A \,|\, \bigcup_{g\in G} A_g = S} \alpha(A,c)^{-1}\alpha(A,k) = \begin{cases} |G|^k, & \text{if } c = k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $c = k$, then the sum is simply equal to the number of possible colorings of d, which is $|G|^k$.

If $c \neq k$, then there exists $s \in S$ such that $c(s) \neq k(s)$. Consider the G-partition A with $\bigcup_{g\in G} A_g = S \setminus \{s\}$, and for $g \in G$, let ${}_gA$ be the G-partition of $\{1, \ldots, n\}$ with ${}_gA_h = A_h$ for all $h \neq g$ and ${}_gA_g = A_g \cup \{s\}$. Then,

$$\begin{aligned}\sum_{g\in G} \alpha({}_gA,c)^{-1}\alpha({}_gA,k) &= \alpha(A,c)^{-1}\alpha(A,k) \sum_{g\in G} \left(\left(\prod_{1\le j\le m} \left(\zeta_{q_i}\right)^{g_j c(s)_j} \right)^{-1} \left(\prod_{1\le j\le m} \left(\zeta_{q_i}\right)^{g_j k(s)_j} \right) \right) \\ &= \alpha(A,c)^{-1}\alpha(A,k) \sum_{g\in G} \left(\prod_{1\le j\le m} \left(\zeta_{q_i}\right)^{g_j(k(s)_j - c(s)_j)} \right) \\ &= \alpha(A,c)^{-1}\alpha(A,k) \prod_{1\le j\le m} \left(\sum_{g_j=0}^{q_j-1} \left(\zeta_{q_i}\right)^{g_j\left(k(s)_j - c(s)_j\right)} \right).\end{aligned}$$

As $c(s) \neq k(s)$, we have that for some j, $c(s)_j \neq k(s)_j$. Thus,

$$\sum_{g_j=0}^{q_j-1} \left(\zeta_{q_i}\right)^{g_j\left(k(s)_j - c(s)_j\right)} = \frac{\left(\zeta_{q_i}\right)^{q_j\left(k(s)_j - c(s)_j\right)} - 1}{\left(\zeta_{q_i}\right)^{\left(k(s)_j - c(s)_j\right)} - 1} = \frac{1-1}{\left(\zeta_{q_i}\right)^{\left(k(s)_j - c(s)_j\right)} - 1} = 0,$$

as $\left(\zeta_{q_i}\right)^{\left(k(s)_j - c(s)_j\right)} - 1 \neq 0$. This implies the result [2]. $\square$

We may now proceed with a proof of the claim that the y_A^T's form a basis for $\mathbb{C}P_n(G)$.

Theorem 3.12.

The set

$$\left\{ y_A^T \mid A \text{ is a } G\text{-partition of } \{1, \ldots, n\} \right\}$$

is a basis for $\mathbb{C}P_n(G)$ *[2].*

Proof. Consider

$$\left| \left\{ y_A^T \mid A \text{ is a } G\text{-partition of } \{1, \ldots, n\} \right\} \right| = \sum_{k=0}^{n} |G|^k \binom{n}{k}^2.$$

This comes from the fact that there is a bijection between the y_A^T set and the set of all colored planar rook diagrams, given by associating a given y_A^T with the diagram d where $\beta(d) = \bigcup_{g \in G} A_g$ and $\tau(d) = T$ such that the color on the edge of d incident to $a \in \beta(d)$ is the color indexing the element of A that contains a.

Let A be a G-partition of $\{1, \ldots, n\}$. Define

$$Y_A^n := \mathbb{C}\text{-span} \left\{ y_A^T \mid T \subseteq \{1, \ldots, n\}, |T| = \left| \bigcup_{g \in G} A_g \right| \right\}.$$

To show that the y_A^T's form a basis for $\mathbb{C}P_n(G)$, we can show that given $d_c \in P_n(G)$, we have

$$d_c \in \sum_A Y_A^n.$$

We proceed by induction on rank(d).

Base Case: If rank(d) = 0, then $d_c = y_A^{\emptyset}$ with $A_g = \emptyset$ for all $g \in G$.

Inductive Step: Suppose that the claim holds for all ranks less than K.

Consider $d_c \in P_n(G)$ with $d : S \to T$ such that $\text{rank}(d) = K$. We now look at

$$\sum_{A \mid \bigcup_{g\in G} A_g = S} \alpha(A,c)^{-1} y_A^T.$$

By Lemma 3.10, we have that

$$\sum_{A \mid \bigcup_{g\in G} A_g = S} \alpha(A,c)^{-1} y_A^T = \sum_{A \mid \bigcup_{g\in G} A_g = S} \alpha(A,c)^{-1} \left(\sum_{k\in \text{Col}(d)} \alpha(A,k) d_k - \sum_{B<A} \left((|G|)^{|A_0/B_0|} y_B^{d(\cup_{g\in G} B_g)} \right) \right)$$

$$= \sum_{A \mid \bigcup_{g\in G} A_g = S} \left(\sum_{k\in \text{Col}(d)} \alpha(A,c)^{-1} \alpha(A,k) d_k \right) - \sum_{A \mid \bigcup_{g\in G} A_g = S} \left(\sum_{B<A} \left(\alpha(A,c)^{-1} (|G|)^{|A_0/B_0|} y_B^{d(\cup_{g\in G} B_g)} \right) \right).$$

By Lemma 3.11, we have that the coefficient on d_k is nonzero if and only if $c = k$. Hence,

$$\sum_{A \mid \bigcup_{g\in G} A_g = S} \left(\sum_{k\in \text{Col}(d)} \alpha(A,c)^{-1} \alpha(A,k) d_k \right) = |G|^k d_k.$$

Additionally, the rank of any diagram in the sum

$$\Xi := \sum_{A \mid \bigcup_{g\in G} A_g = S} \alpha(A,c)^{-1} y_A^T - |G|^k d_k$$

must be less than K. By our inductive hypothesis, $\Xi \in \sum_A Y_A^n$. Thus, $\sum_{A \mid \bigcup_{g\in G} A_g = S} \alpha(A,c)^{-1} y_A^T \in \sum_A Y_A^n$. This implies that $d_c \in \sum_A Y_A^n$, as all other terms may be canceled by undoing nonzero multiplication by $\alpha(A,c)^{-1}$'s and subtracting diagrams with rank less than K (all diagrams in Ξ other than d_c satisfy this criterion). $\square$

Theorem 3.13.

Each Y_A^n is $P_n(G)$-invariant and irreducible [2].

Proof. Theorem 3.9 tells us that the action of $P_n(G)$ on a basis element of Y_A^n produces either a scalar multiple of another basis element of Y^n_A or $\vec{0}$. Thus, Y^n_A is $P_n(G)$-invariant.

For the sake of contradiction, suppose W is a non-trivial proper P_n-invariant subspace of Y^n_A. Let

$$\vec{0} \neq \vec{\gamma} = \sum_{T \,|\, |T| = |\bigcup_{g \in G} A_g|} \lambda_T y^T_A$$

be a nonzero element of W. Since $\vec{\gamma}$ is nonzero, there exists some S such that λ_S is nonzero. Let $d_c \in P_n(G)$ with $d : S \to S$ and $c(s) = e$ for all $s \in S$. Then,

$$(\vec{\gamma})^{d_c} = \lambda_S y^S_A.$$

We can undo the multiplication by nonzero λ_S to see that $y^S_A \in W$.

Let y^X_A be a basis element of Y^n_A. Let $b_k \in P_n(G)$ with $b : S \to X$ and $k(s) = e$ for all $s \in S$. Consider

$$\left(y^S_A\right)^{b_k} = y^{b(S)}_A = y^X_A.$$

Hence, $y^X_A \in W$

Thus, any one basis vector generates the entire space Y^n_A. Hence, $W = Y^n_A$, a contradiction.

Therefore, each Y^n_A is P_n-invariant and irreducible. □

Theorem 3.14.

For G a finite abelian group, the colored planar rook algebra $\mathbb{C}P_n(G)$ is semisimple and decomposes as

$$\mathbb{C}P_n(G) = \bigoplus_{A \text{ is a } G\text{-partition of } \{1,\ldots,n\}} Y^n_A.$$

Proof. The regular representation of $\mathbb{C}P_n(G)$ is completely reducible, as we have constructed the appropriate decomposition. By the Artin-Wedderburn Theorem (Theorem 1.6), we have that $\mathbb{C}P_n(G)$ is semisimple [2]. □

We have done in this section what we set out to do. The colored planar rook algebra $\mathbb{C}P_n(G)$ is semisimple for finite abelian G.

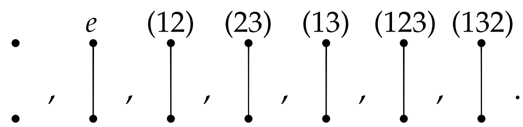
We will see in the next section (Section 3.3.3) that dimension plays a role in constructing a counterexample to the claim that $\mathbb{C}P_n(G)$ is semisimple for all G.

3.3.3 $\mathbb{C}P_n(G)$ For Finite Non-Abelian G

That $\mathbb{C}P_n(G)$ is semisimple for finite abelian G does not extend to finite non-abelian G. In this section, we exemplify this unfortunate fact.

Example 3.12. The colored planar rook algebra $\mathbb{C}P_1(S_3)$ is not semisimple.

The elements of $P_1(S_3)$ are

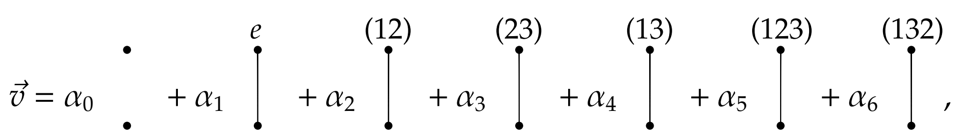


If

$$\mathbb{C}P_1(S_3) = \bigoplus_{i \in I} V_i$$

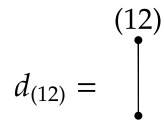
is a decomposition of $\mathbb{C}P_1(S_3)$ into irreducible $\mathbb{C}P_1(S_3)$-modules, then Theorem 1.6 (2) tells us that the sum of the squares of the dimensions of the V_i's is equal to the dimension of $\mathbb{C}P_1(S_3) = 7$. The only two decompositions of 7 into sums of squares (of positive integers) are $7 = 2^2 + 1 + 1 + 1$ and $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$. In either case, there are at least three 1-dimensional representations of $\mathbb{C}P_1(S_3)$. We will show that, in fact, there exist exactly two 1-dimensional representations of $\mathbb{C}P_1(S_3)$.

Suppose

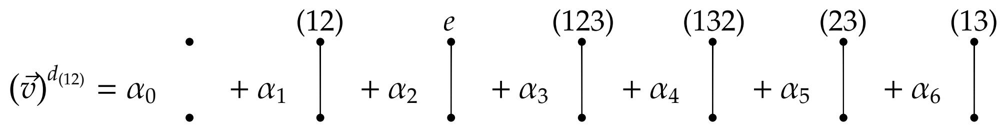


where $\alpha_i \in \mathbb{C}$, generates a 1-dimensional $P_1(S_3)$-invariant irreducible $\mathbb{C}P_1(S_3)$-module.

Let



act on $\vec{v}$. Then,



As $\vec{v}$ generates a P_n-invariant submodule, we have that $(\vec{v})^{d_{(12)}}$ is a scalar multiple of $\vec{v}$. That is, $(\vec{v})^{d_{(12)}} = \lambda\vec{v}$ for some $\lambda \in \mathbb{C}$. Then,

$$\alpha_0 = \lambda\alpha_0;$$

$$\alpha_1 = \lambda\alpha_2;$$

$$\alpha_2 = \lambda\alpha_1;$$

$$\alpha_3 = \lambda\alpha_5;$$

$$\alpha_4 = \lambda\alpha_6;$$

$$\alpha_5 = \lambda\alpha_3;$$

$$\alpha_6 = \lambda\alpha_4.$$

From this, we see that

$$\alpha_0 = \lambda\alpha_0;$$

$$\alpha_1 = \lambda\alpha_2 = \lambda^2\alpha_1;$$

$$\alpha_3 = \lambda\alpha_5 = \lambda^2\alpha_3;$$

$$\alpha_4 = \lambda\alpha_6 = \lambda^2\alpha_4.$$

Hence, $\alpha_i = 0$ for all $i > 0$ or $\lambda = 1$. Now consider

$$(\vec{v})^{d_{(23)}} = \alpha_0 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} + \alpha_1 \overset{(23)}{\Big|} + \alpha_2 \overset{(132)}{\Big|} + \alpha_3 \overset{e}{\Big|} + \alpha_4 \overset{(123)}{\Big|} + \alpha_5 \overset{(13)}{\Big|} + \alpha_6 \overset{(12)}{\Big|}$$

Again, we have a scalar multiple of $\vec{v}$. So,

$$\alpha_0 = \omega\alpha_0;$$

$$\alpha_1 = \omega\alpha_3;$$

$$\alpha_2 = \omega\alpha_6;$$

$$\alpha_3 = \omega\alpha_1;$$

$$\alpha_4 = \omega\alpha_5;$$

$$\alpha_5 = \omega\alpha_4;$$

$$\alpha_6 = \omega\alpha_2,$$

and we see that $\alpha_i = 0$ for all $i > 0$ or $\omega = 1$.

If $\alpha_i = 0$ for all $i > 0$, then α_0 is nonzero, as $\vec{v}$ was assumed to be nonzero. Thus, the representation is the set of scalar multiples of the empty diagram.

Suppose $\alpha_i \neq 0$ for some $i > 0$. Then, $\lambda = \omega = 1$, implying that all α_i for $i > 0$ are equal. Then, if d_0 is the empty diagram,

$$(\vec{v})^{d_0} = \alpha_0 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} + \alpha_1 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} + \alpha_2 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} + \alpha_3 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} + \alpha_4 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} + \alpha_5 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} + \alpha_6 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix}$$

$$= \alpha_0 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} + 6\alpha_1 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} .$$

This is a scalar multiple of $\vec{v}$, as $\vec{v}$ generates the entire representation. As $(\vec{v})^{d_0}$ has no non-empty diagrams, it must be the case that $(\vec{v})^{d_0} = \vec{0}$. Then,

$$\alpha_0 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} + 6\alpha_1 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} = \vec{0}$$

implies that

$$\alpha_0 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} = -6\alpha_1 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} .$$

Thus, $\alpha_0 = -6\alpha_1$. Then, the representation generated by $\vec{v}$ is

$$\mathbb{C}\text{-span}\left\{-6 \begin{smallmatrix}\bullet\\ \\ \bullet\end{smallmatrix} + \overset{e}{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} + \overset{(12)}{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} + \overset{(23)}{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} + \overset{(13)}{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} + \overset{(123)}{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}} + \overset{(132)}{\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}}\right\}.$$

This is a 1-dimensional representation of $\mathbb{C}P_1(S_3)$. The only other 1-dimensional representation of $\mathbb{C}P_1(S_3)$ is the set of scalar multiples of the empty diagram. That is, we have exhausted all possibilities, implying that there are exactly two distinct 1-dimensional representations of $\mathbb{C}P_1(S_3)$. Thus, $\mathbb{C}P_1(S_3)$ is not semisimple.

This example is the only concrete one we have of the colored planar rook algebra not being semisimple. In our concluding chapter, we will make some observations about the lack of semisimplicity of $\mathbb{C}P_1(S_3)$ and discuss some potential implications.

CHAPTER 4

Conclusion

In the reasoning that $\mathbb{C}P_1(S_3)$ is not semisimple, one key component is the fact that the sets of equations we receive upon acting on a linear combination of diagrams do not behave "nicely." We now attempt to generalize these sets of equations.

Definition 4.1 Let G be a group and $g \in G$. Then,

$$G^g = \{(x, x^g) \mid x \in G\}$$

is called the **g-hit** of G.

Example 4.1. Consider $(23) \in S_3$. We can look at the (23)-hit of S_3:

$$\begin{array}{ccccccc} & e & (12) & (23) & (13) & (123) & (132) \\ (23) & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & (23) & (132) & e & (123) & (13) & (12) \end{array}$$

So,

$$S_3^{(23)} = \{(e,(23)),((12),(132)),((23),e),((13),(123)),((123),(13)),((132),(12))\}$$

Definition 4.2 Let G be a group and $g \in G$. Then,

$$xG^g = \left\{y \in G^g \mid y = x^{(g^k)} \text{ for some } k \in \mathbb{N}\right\},$$

is called the x**-block** of G^g.

Example 4.2. Taking Example 4.1, we can look at the x-blocks of $S_3^{(23)}$ for $x \in S_3$. They are

$$\begin{aligned}
eS_3^{(23)} &= \{e, (23)\} \\
(12)S_3^{(23)} &= \{(12)(132)\} \\
(23)S_3^{(23)} &= \{e, (23)\} \\
(13)S_3^{(23)} &= \{(13), (123)\} \\
(123)S_3^{(23)} &= \{(13), (123)\} \\
(132)S_3^{(23)} &= \{(12), (132)\} \,.
\end{aligned}$$

Note that there are 3 distinct x-blocks, each of size 2, and that they partition S_3.

Theorem 4.1.

Let G be a group and $x, g \in G$. Then, $xG^g = \langle g \rangle x$, the coset of $\langle g \rangle$ with representative x.

Proof. Note that $a \in xG^g$ if and only if $a = x^{(g^k)} = g^k x$ for some $k \in \mathbb{N}$. This is true if and only if $a \in \langle g \rangle x$. □

Theorem 4.2.

Let G be a finite group and $g \in G$. For $a, b \in G$, define a relation $\sim$ by $a \sim b$ if and only if for $x \in G$, a and b are in the same x-block of G^g. Then,

1. *The relation $\sim$ is an equivalence relation. Thus, the x-blocks of G^g partition G.*
2. *The size of an x-block of G^g is equal to the order of g. That is, $|xG^g| = o(g)$.*
3. *The number of distinct x-blocks is equal to $|G/\langle g \rangle|$.*

Proof. 1. This is true because $xG^g = \langle g \rangle x$, and the cosets of a subgroup of G partition the group.

2. This follows from the definition of xG^g. The elements of xG^g look like $x^{(g^k)} = g^k x$. Then, $g^m x = g^n x$ if and only if $m \equiv n \mod o(g)$. So, the number of distinct values $x^{(g^k)}$ can assume is equal to $o(g)$.

3. This follows from 1 and 2. As x-blocks partition G and the size of an x-block is $o(g) = |\langle g \rangle|$, we have that $|G/\langle g \rangle| = |G|/|\langle g \rangle| = |G|/o(g)$ is the number of distinct x-blocks.

$\square$

We believe that cosets and normal subgroups are key in the understanding of why $\mathbb{C}P_n(G)$ is semisimple for finite abelian G. All subgroups of an abelian group are normal, so left and right cosets are equal. This seems to cause the sets of equations we discovered in Section 3.3.3 to "work out nicely." When a subgroup is not normal, these equations may not behave in such a manner. The method for decomposing $\mathbb{C}P_n(G)$ for finite abelian G is complicated, and a method for some finite non-abelian G is unknown (if one exists). We have only the example $\mathbb{C}P_1(S_3)$ to show us that there is a group G for which $\mathbb{C}P_n(G)$ is not semisimple. We would like to explore $\mathbb{C}P_1(Q_8)$, as Q_8 is one of the next smallest non-abelian groups (after S_3). Additionally, all of Q_8's subgroups are normal, which can not be said of S_3. We also wish to look at D_4, the other non-abelian group of order 8.

This Independent Study has looked at the planar rook algebra. The thesis itself mirrors the process by which it was written. The project began by looking at the uncolored planar rook algebra $\mathbb{C}P_n$ and reading [1] to discover what it meant for $\mathbb{C}P_n$ to be semisimple. Intrigued by the algebra, we then looked at understanding the semisimple decomposition of $\mathbb{C}P_n(G)$ for finite abelian G ([2]). Again, one future goal is to explore the colored planar rook algebra when colored by Q_8 and D_4. Other future work includes coding the α-function (Definition 3.2), further exploring the subalgebras $\mathbb{C}P_n(G)[S]$ (Theorem 3.4) and finding a simpler method of decomposing $\mathbb{C}P_n(G)$ for finite abelian G.

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